Lecture 17 — Optimal Auctions

Algorithmic Game Theory & Applications (AGTA) - 2025

Guest Lecture: Yiannis Giannakopoulos (University of Glasgow) — 17 March 2025

Single-Item Auctions:
Quick Refresher

Model: Single-Item Auctions

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  \+ Individual rationality (IR): $u_i(v_i, \boldsymbol{b}_{-i}; v_i) \geq 0$ $\forall \boldsymbol{b}\ \forall i\ \forall v_i$

Myerson’s Characterization

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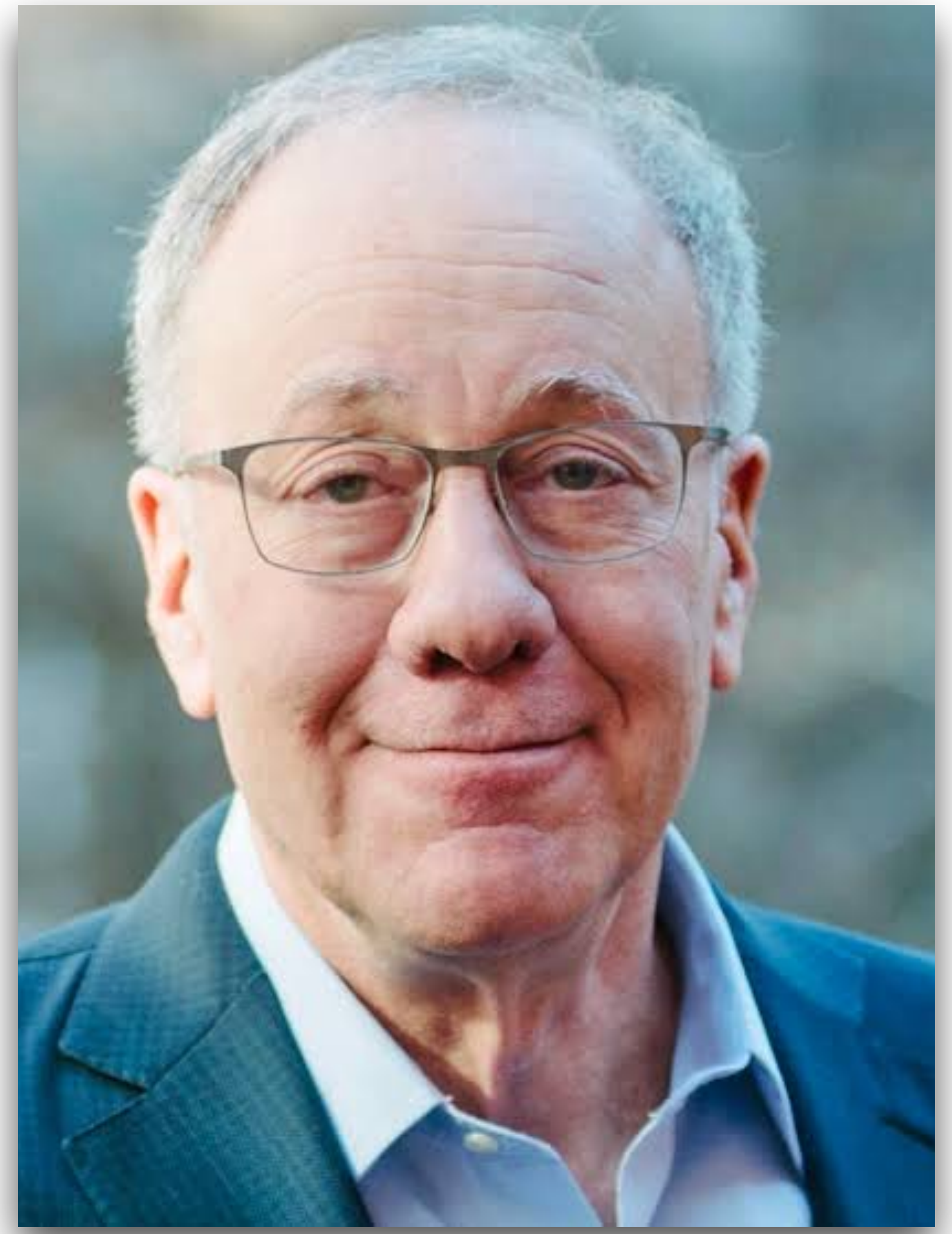

Roger Myerson (1951 -)

Nobel prize in Economics
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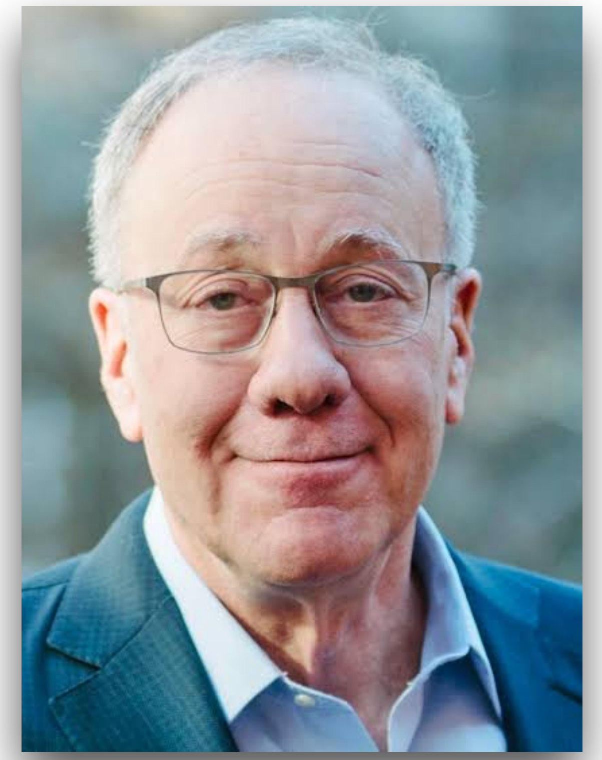

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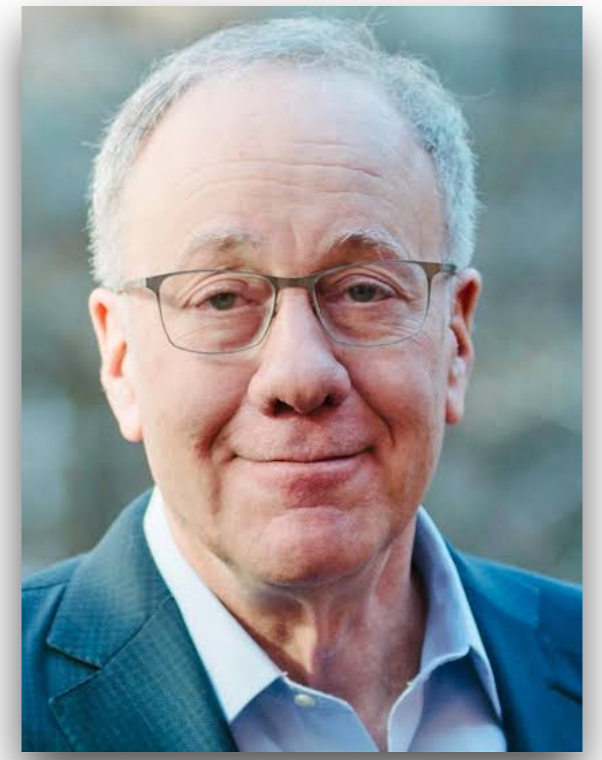

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  - Shall we still *always* sell to the *highest* bidder?

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Welfare vs Revenue

Example: *Single-Bidder Deterministic* Auctions

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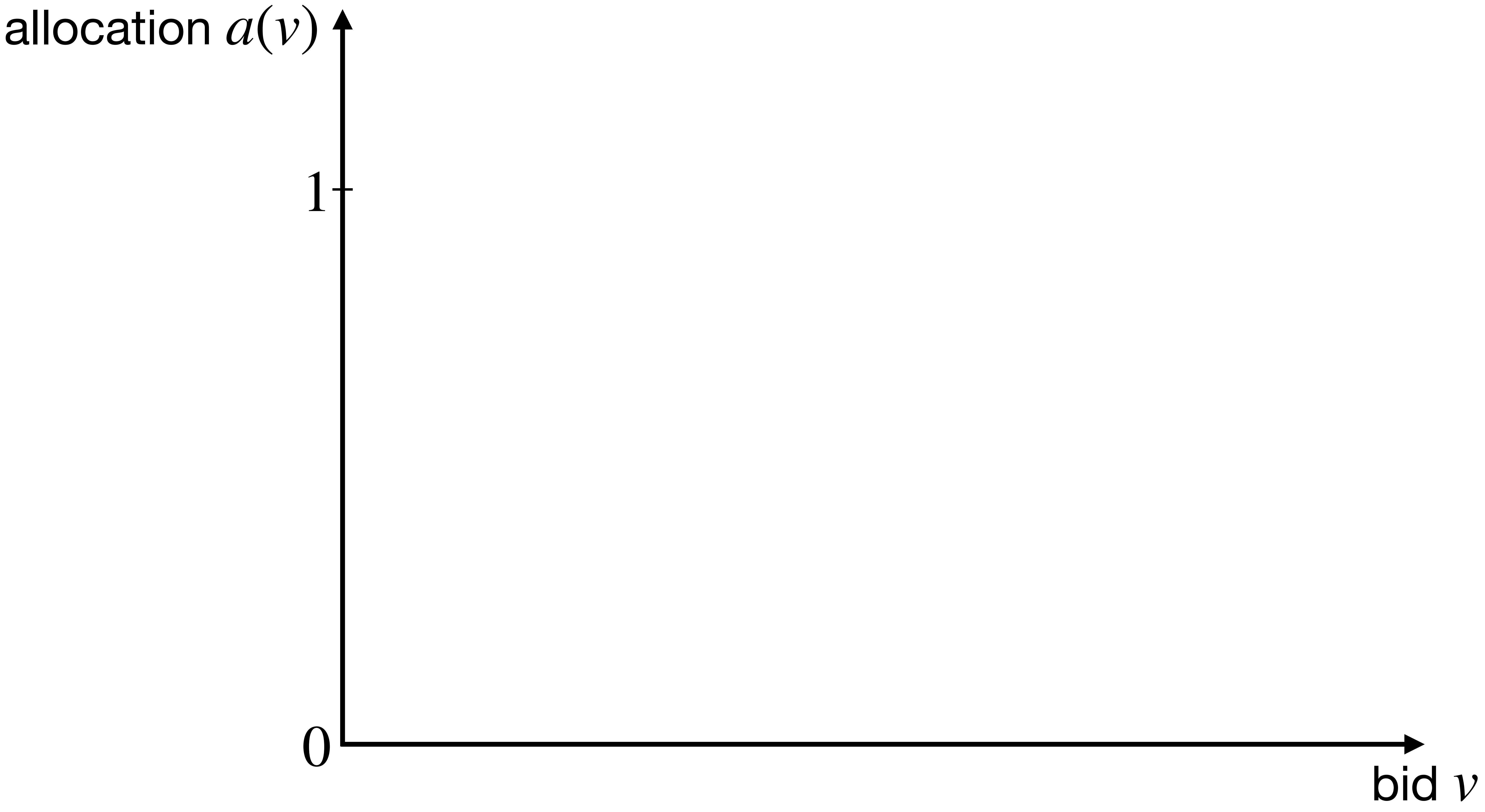

Welfare vs Revenue

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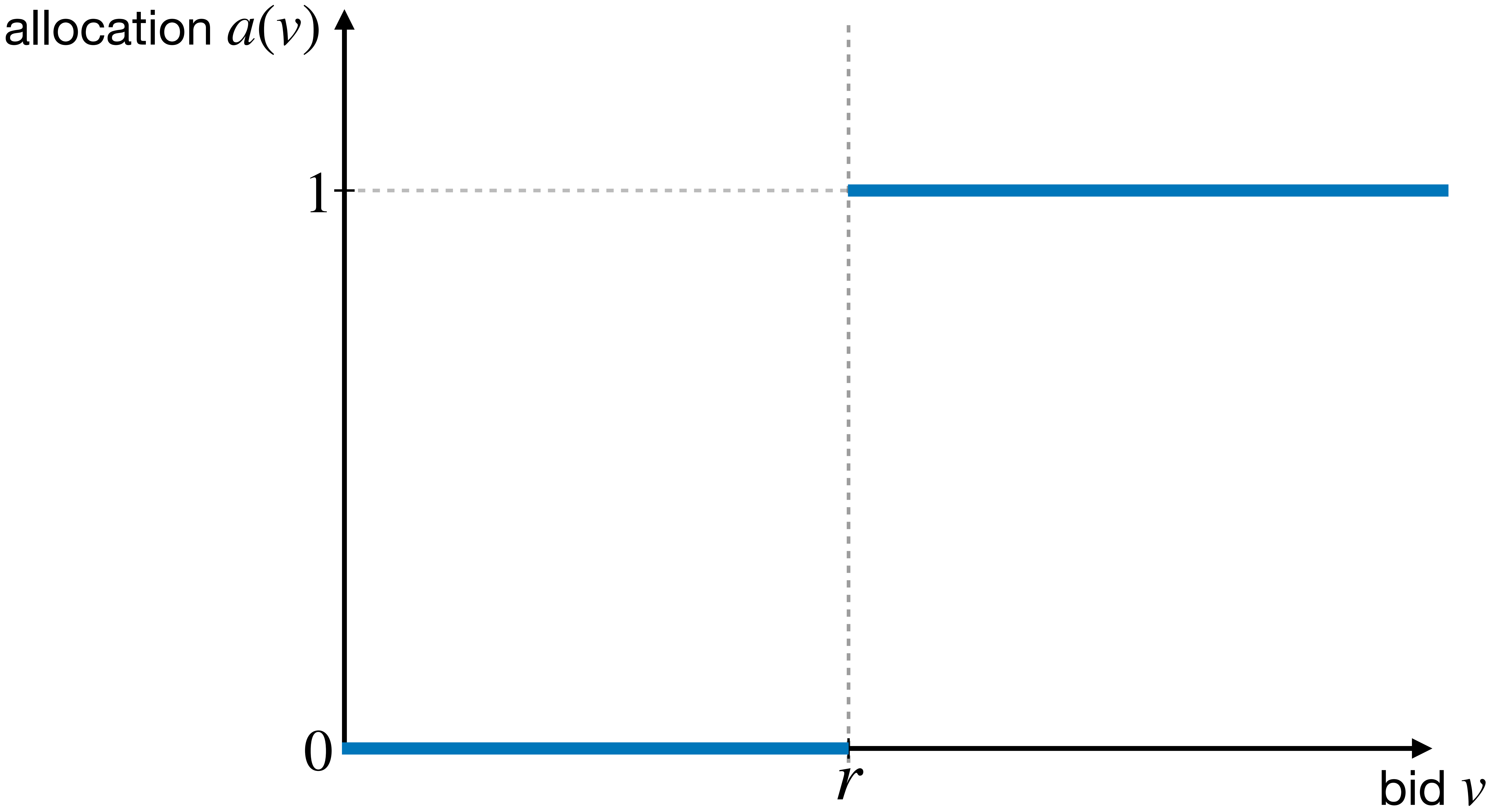

Welfare vs Revenue

Example: *Single-Bidder Deterministic* Auctions

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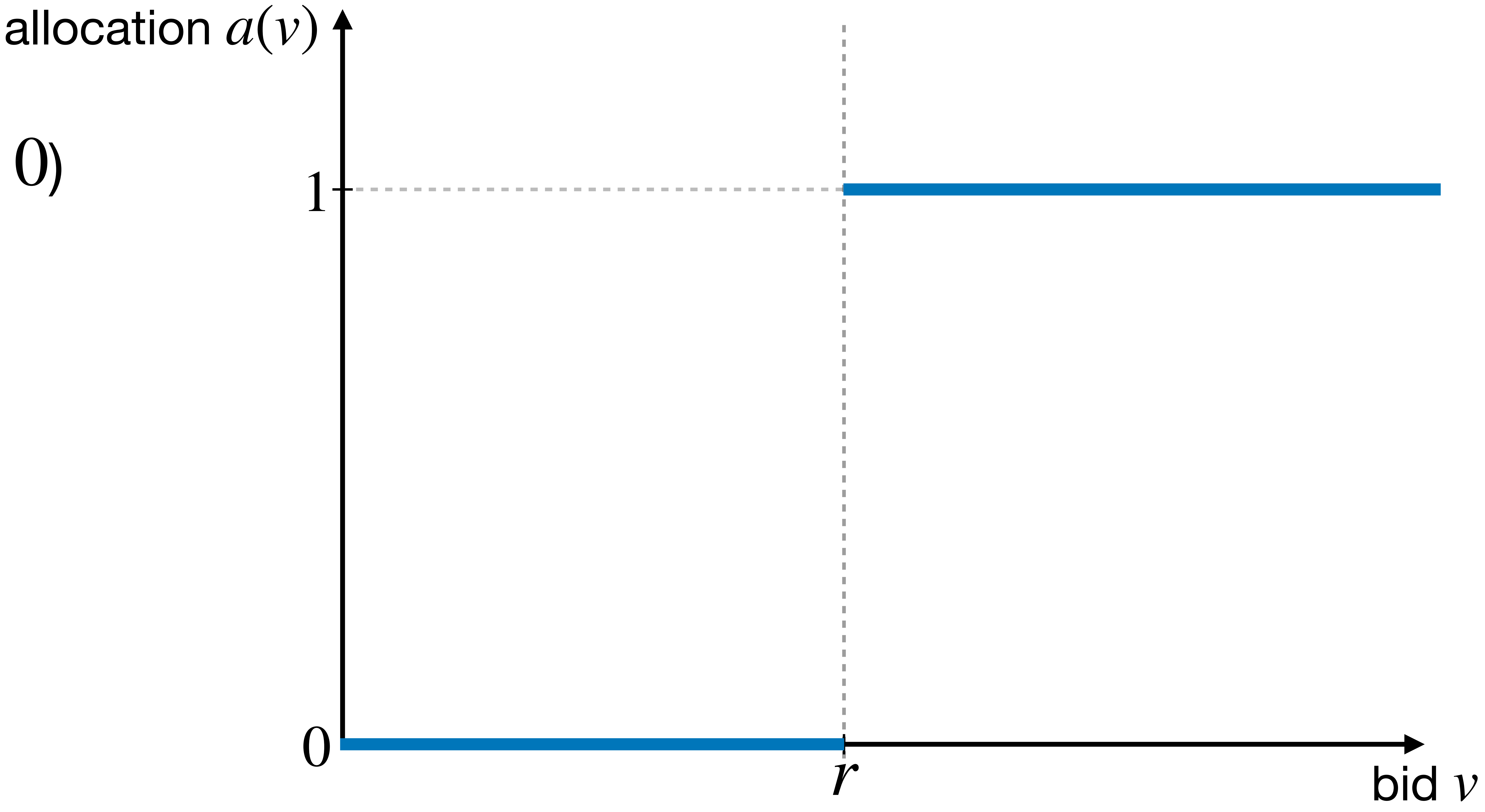

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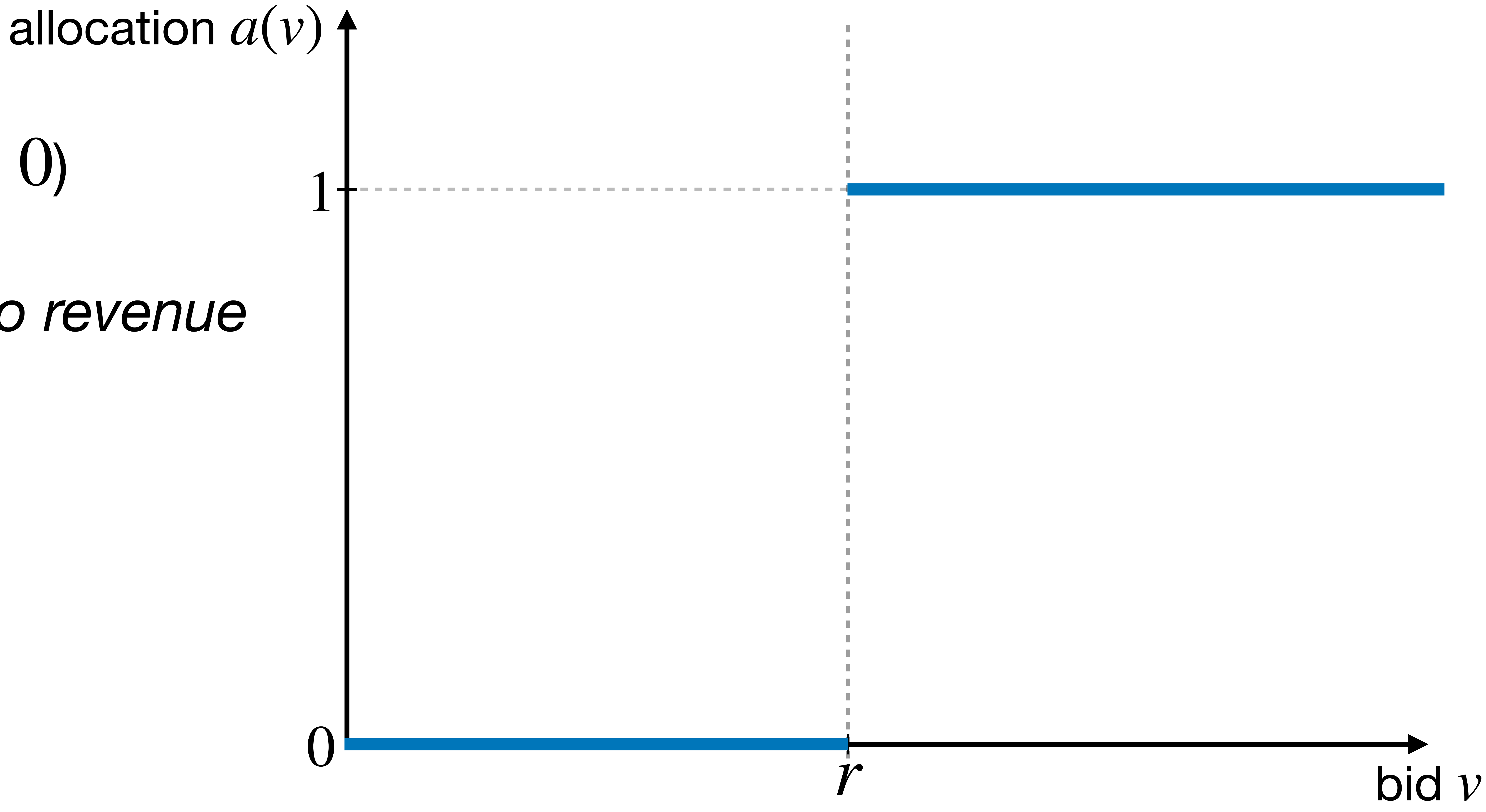

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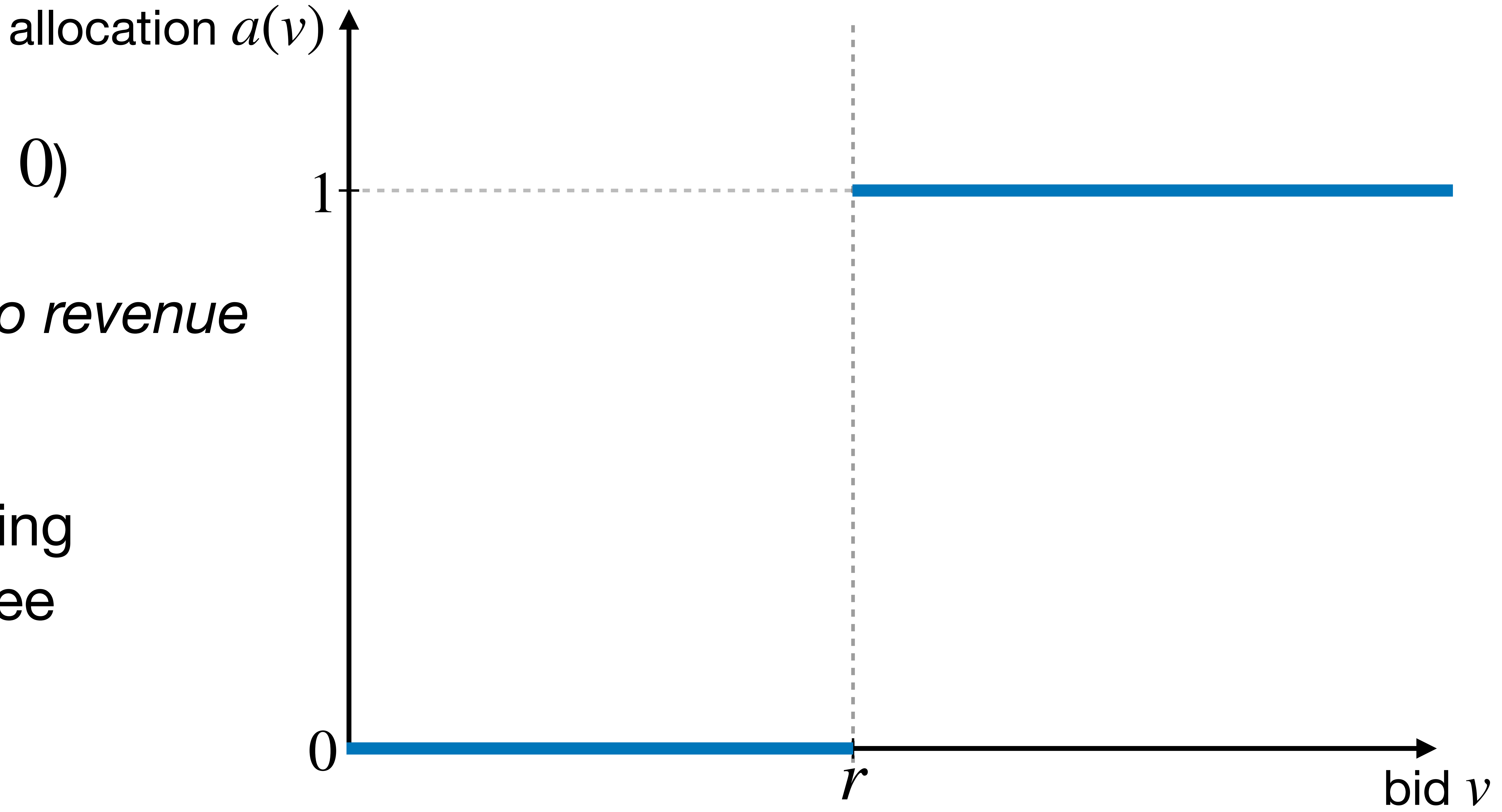

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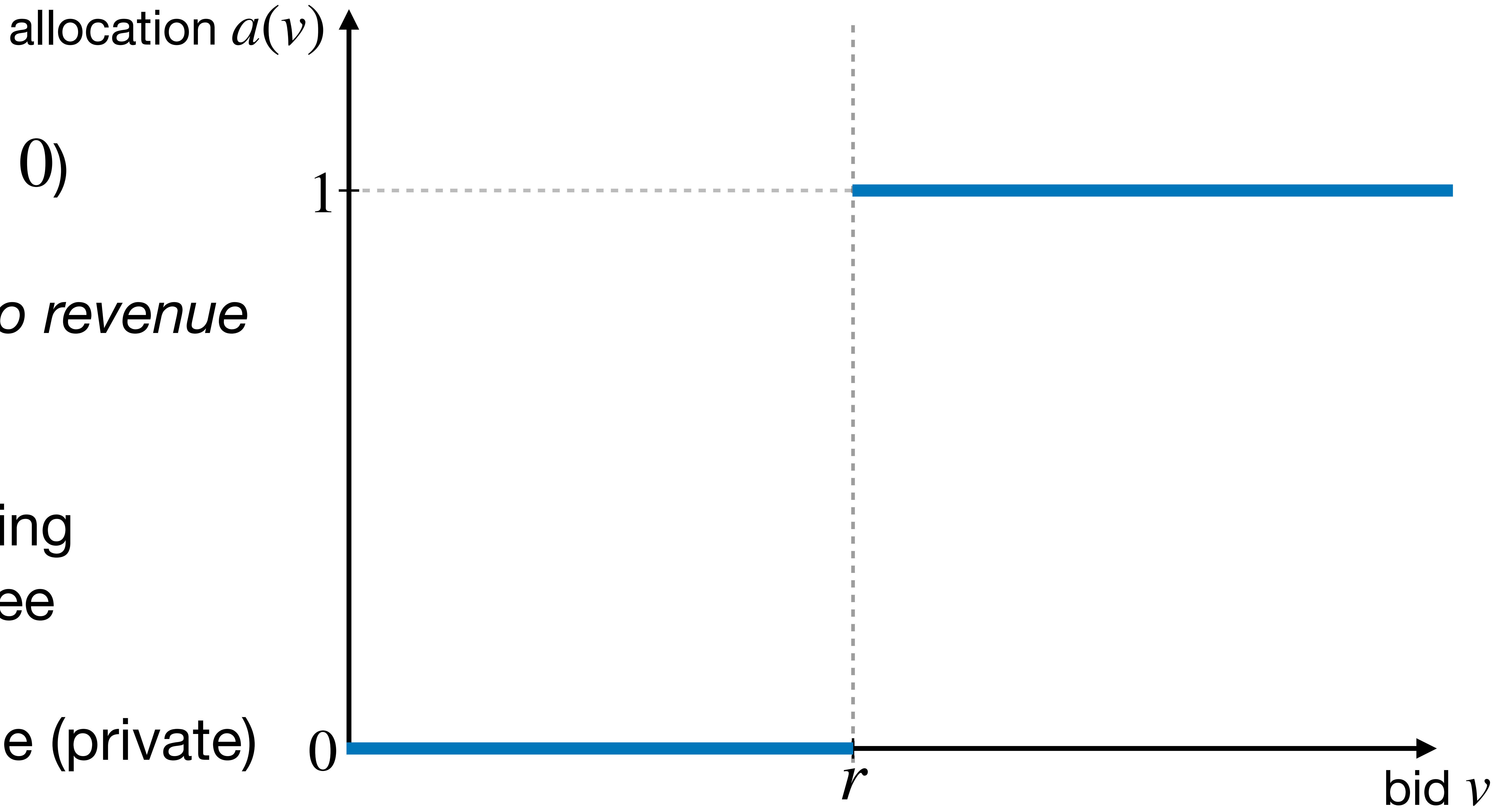
  - *Highly* dependent on the (private) value v of the bidder.



Bayesian (Single-Item) Auctions

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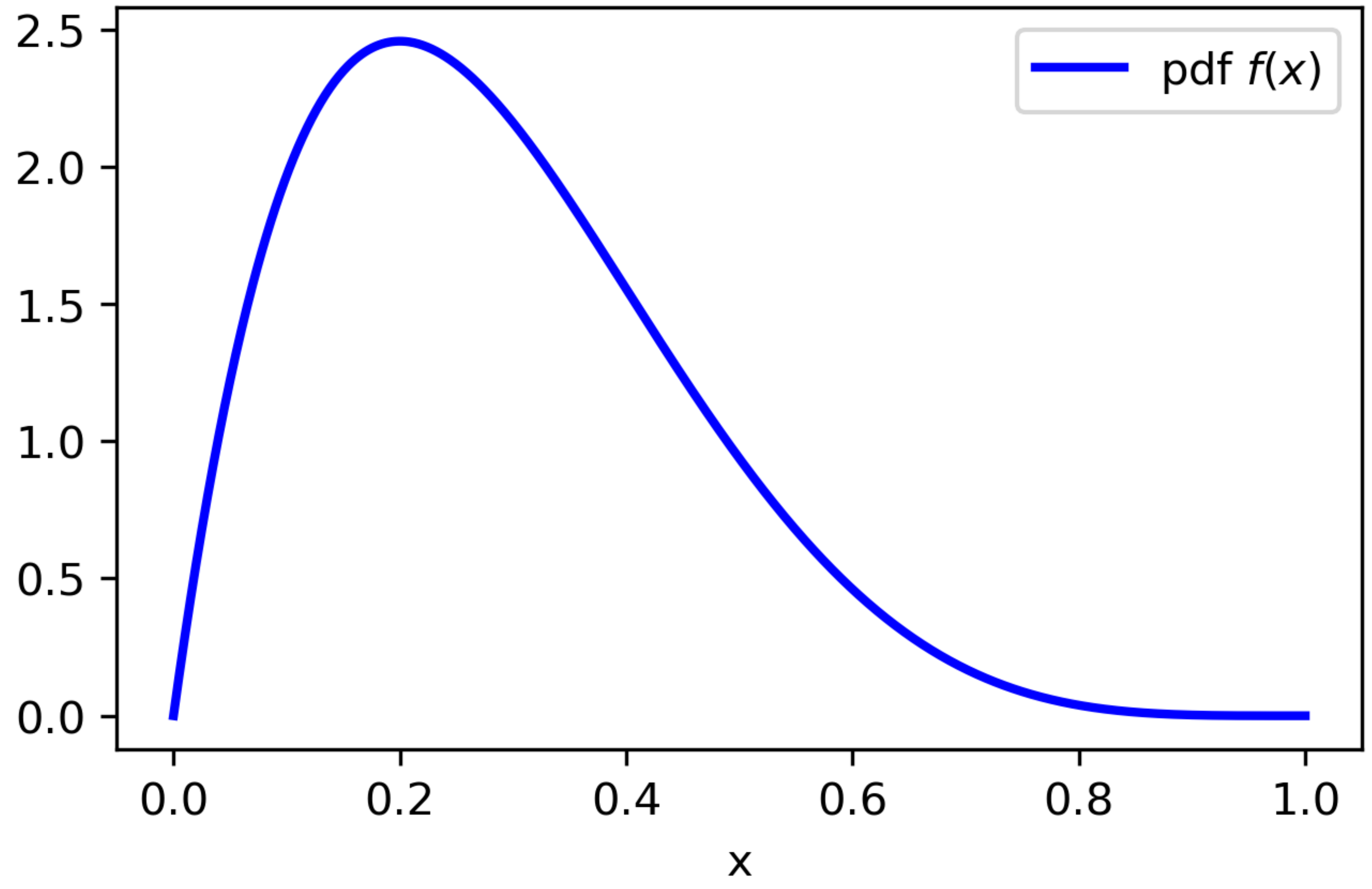

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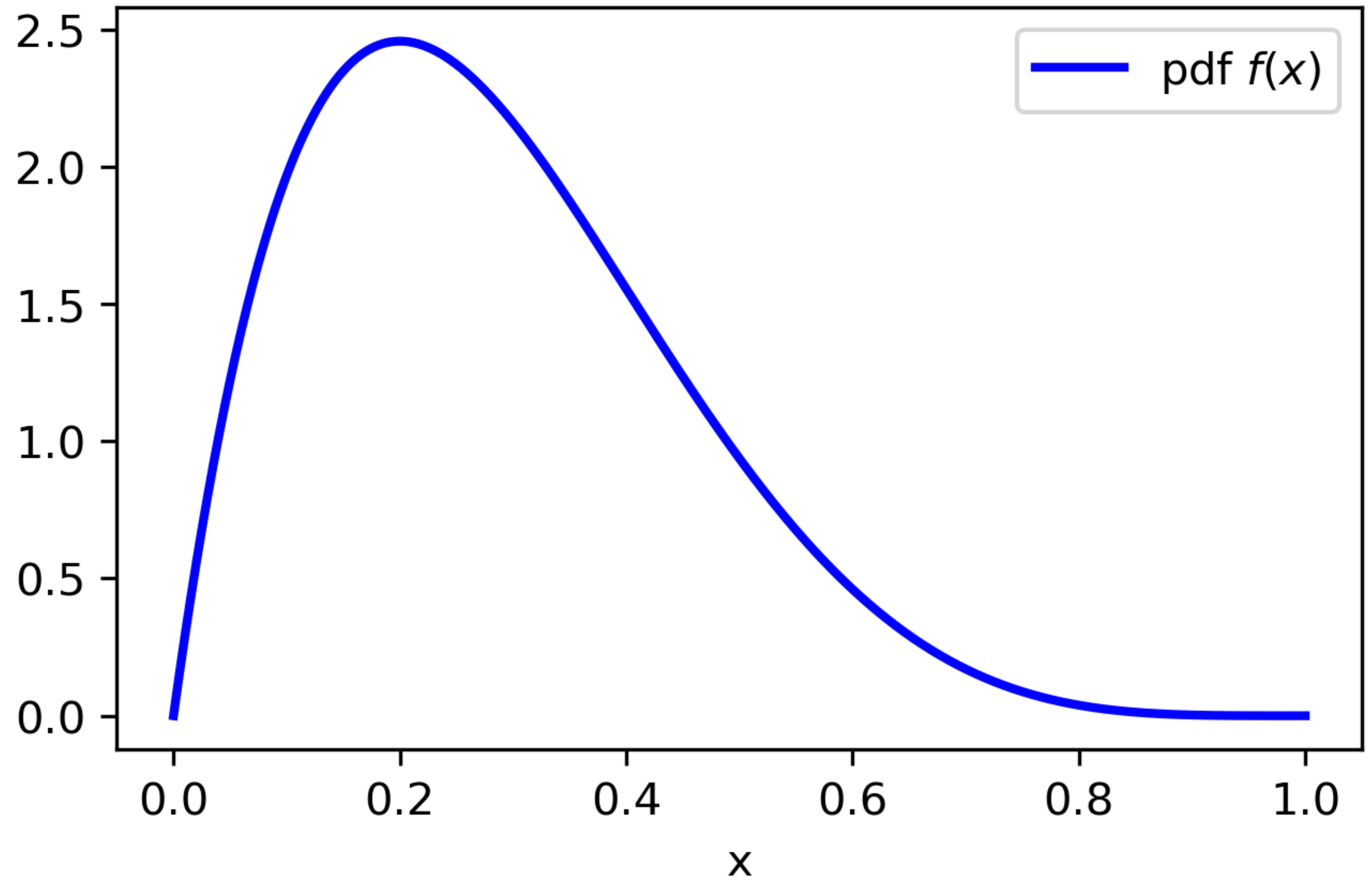

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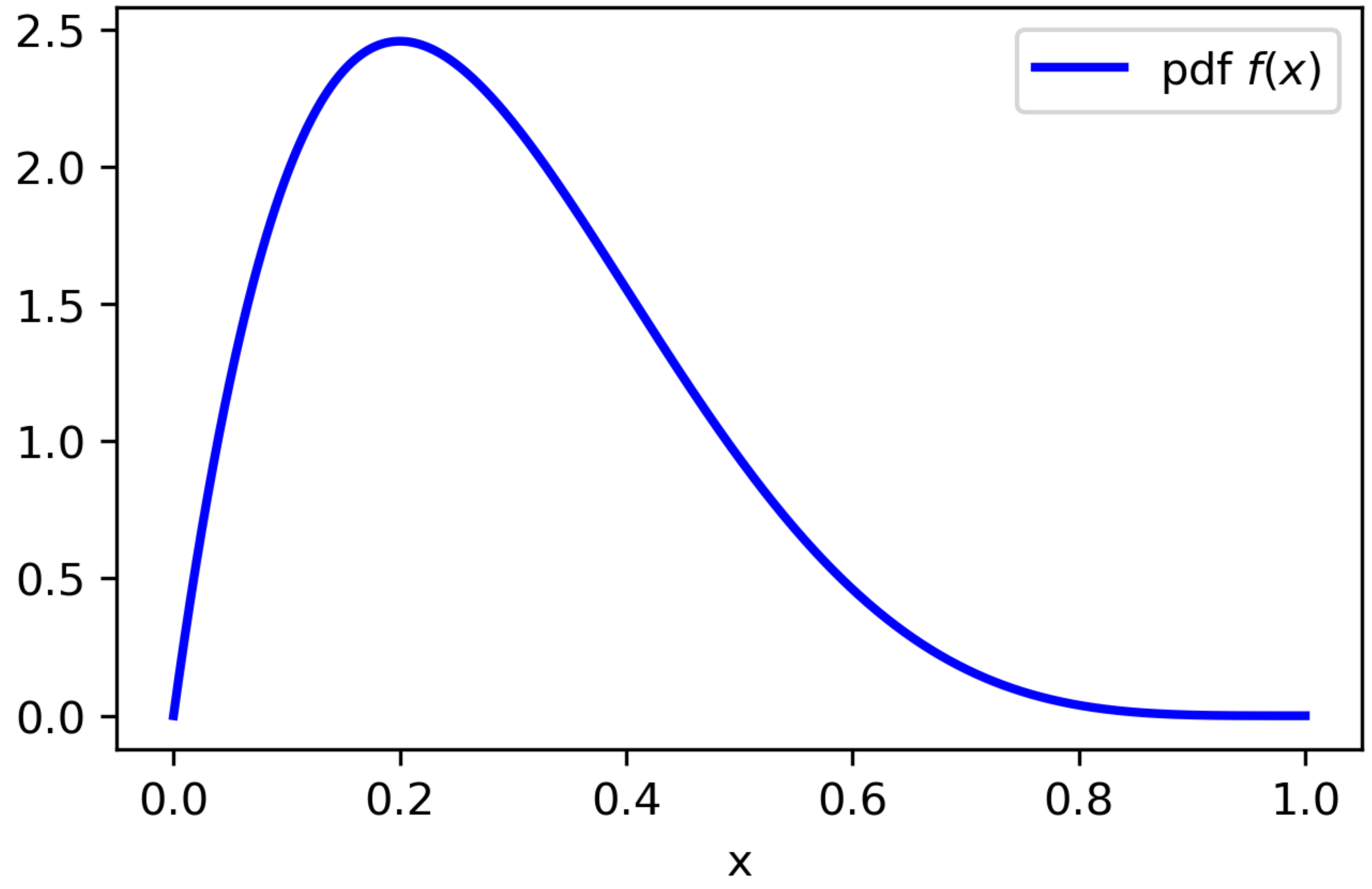

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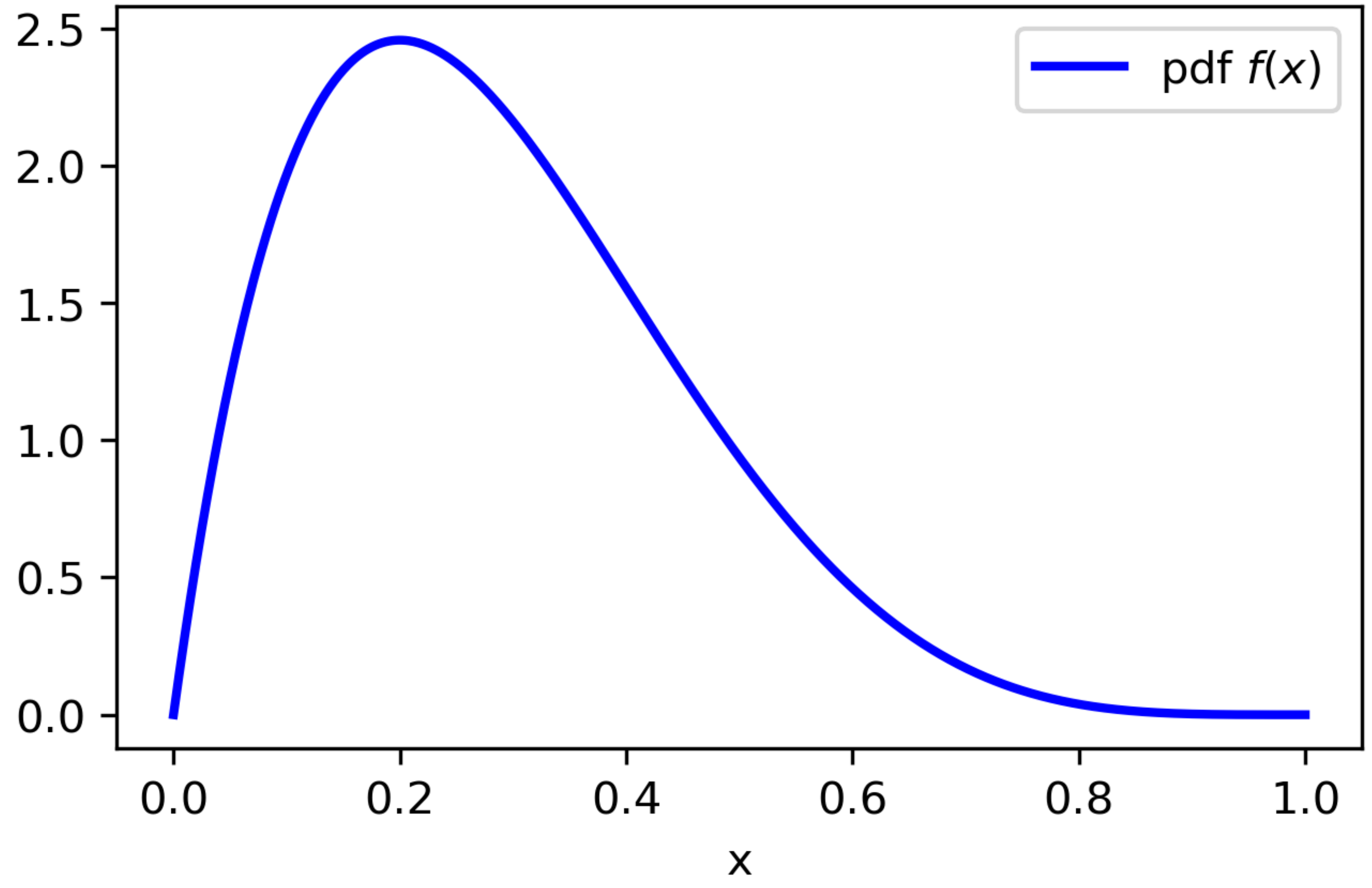

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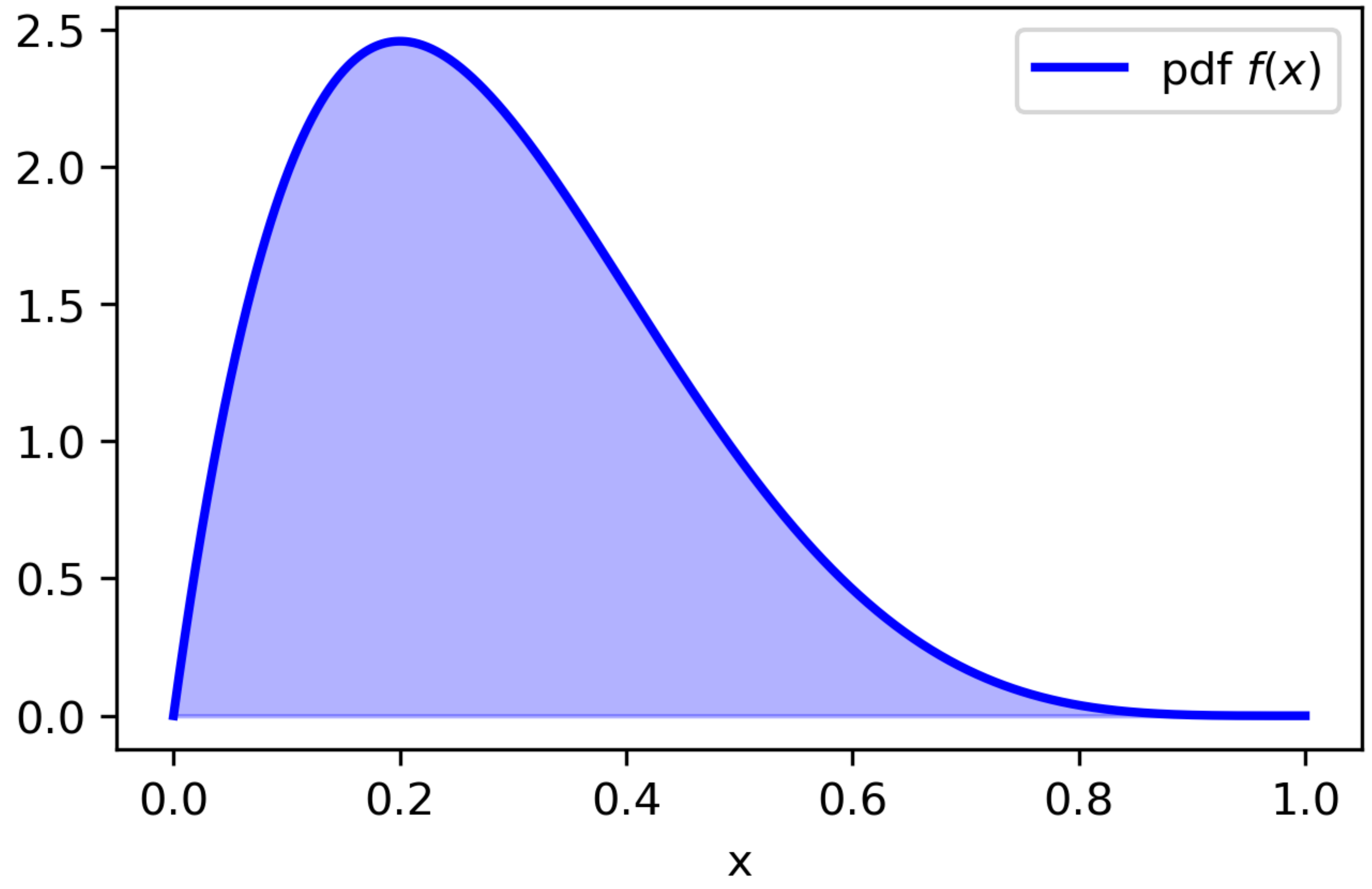

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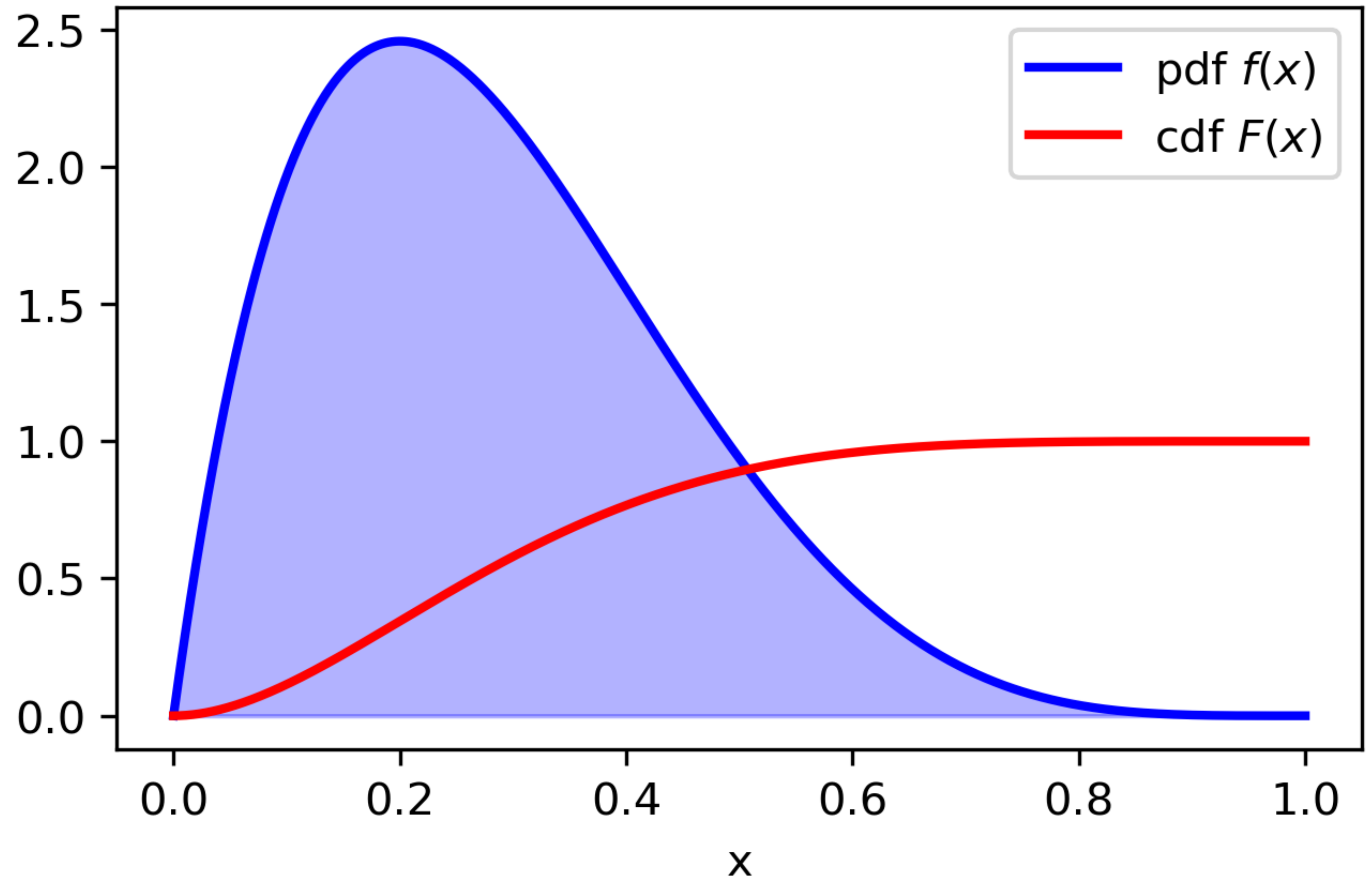

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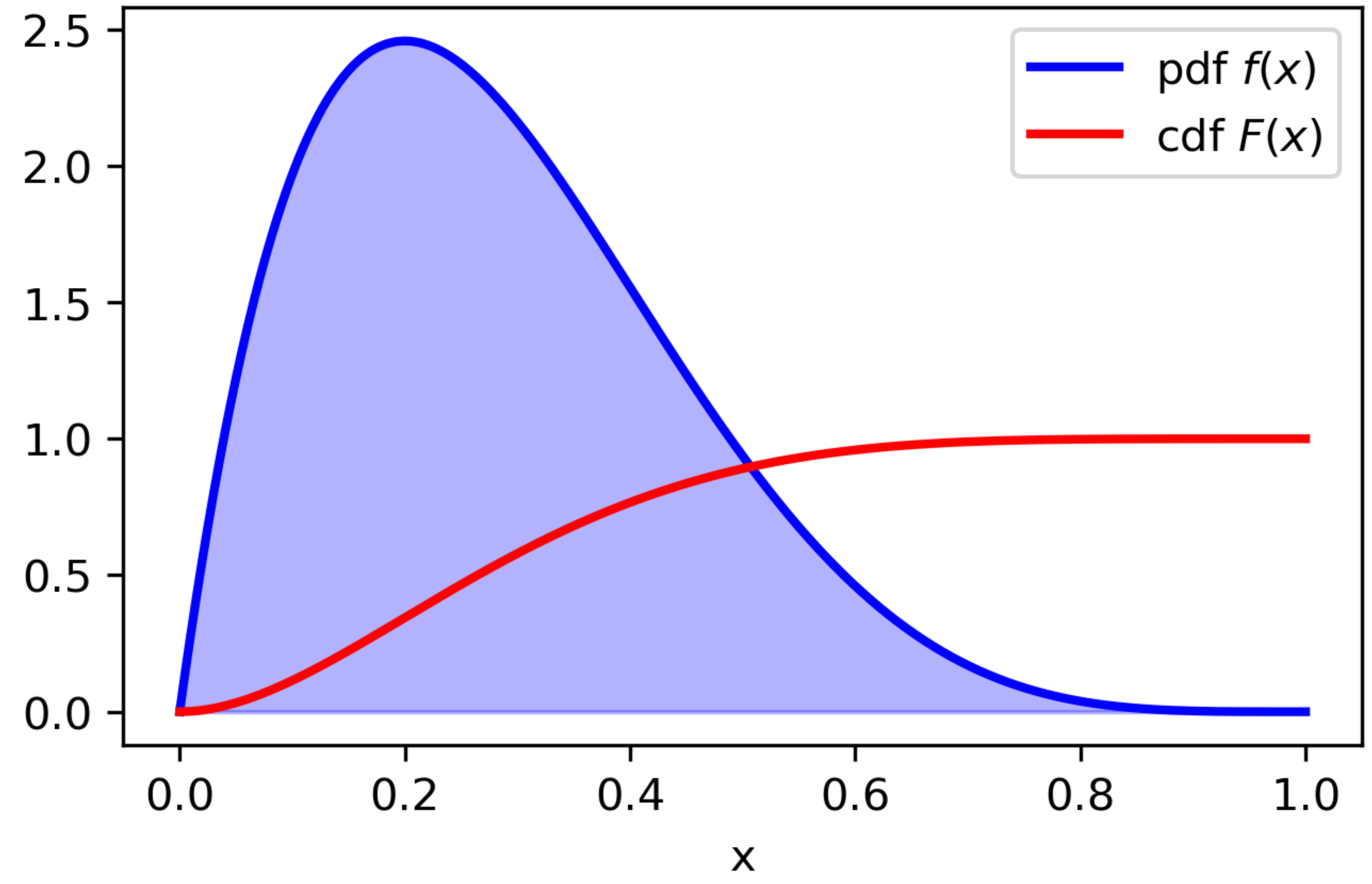

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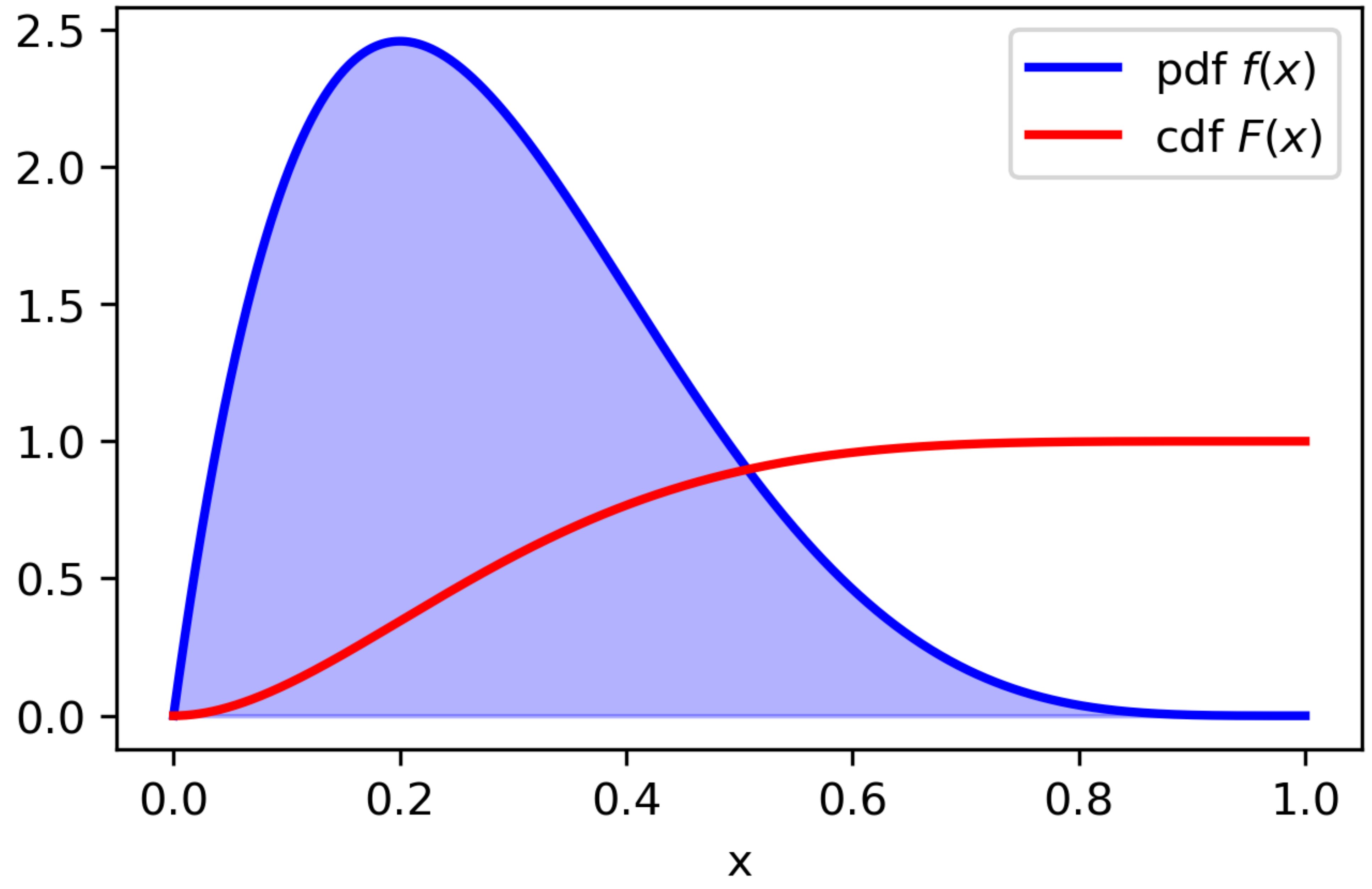

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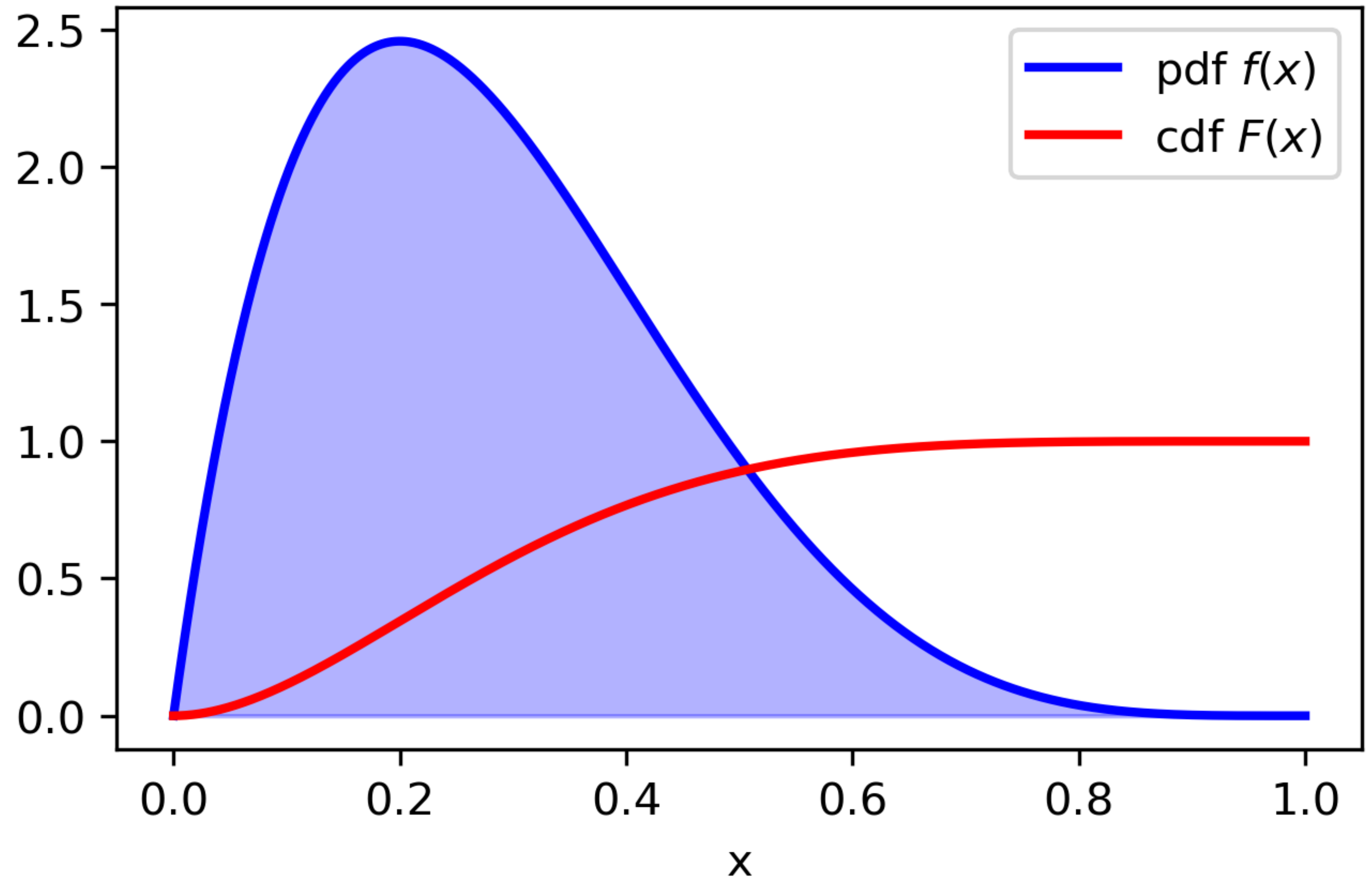

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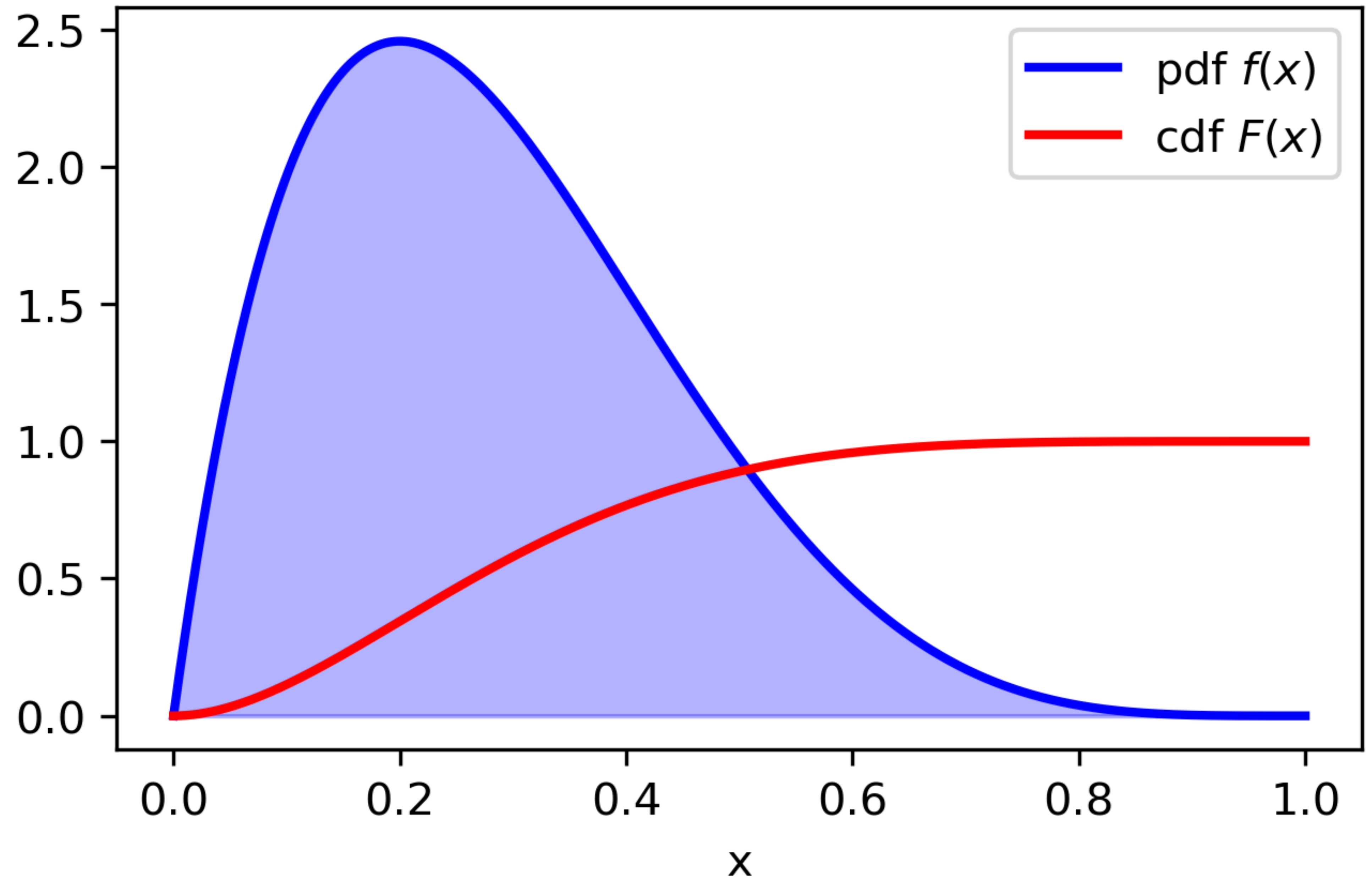

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Example

Uniform distribution on $[0,1]$:

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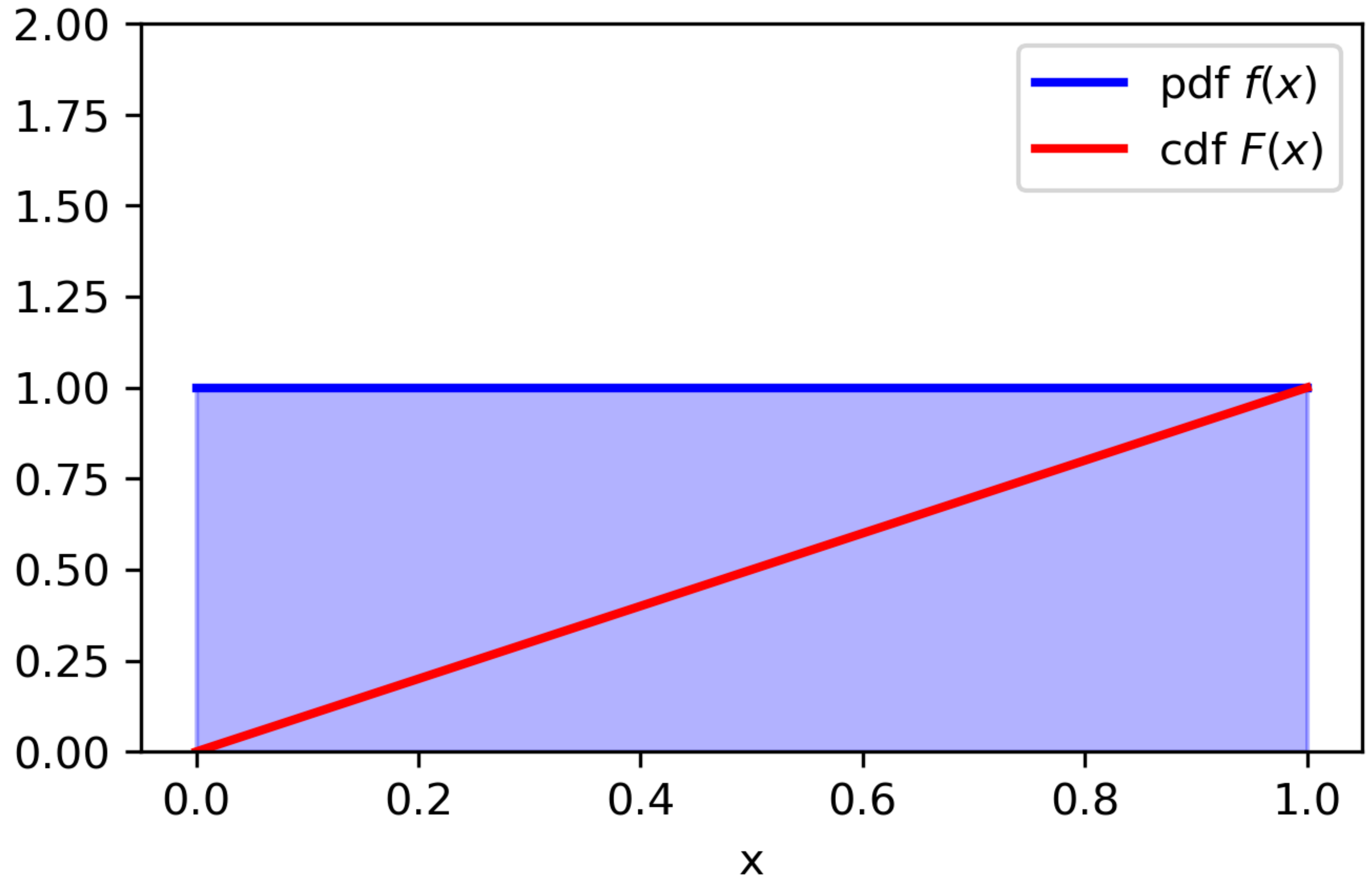

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Revenue

- Goal: find a **revenue-maximizing** truthful auction

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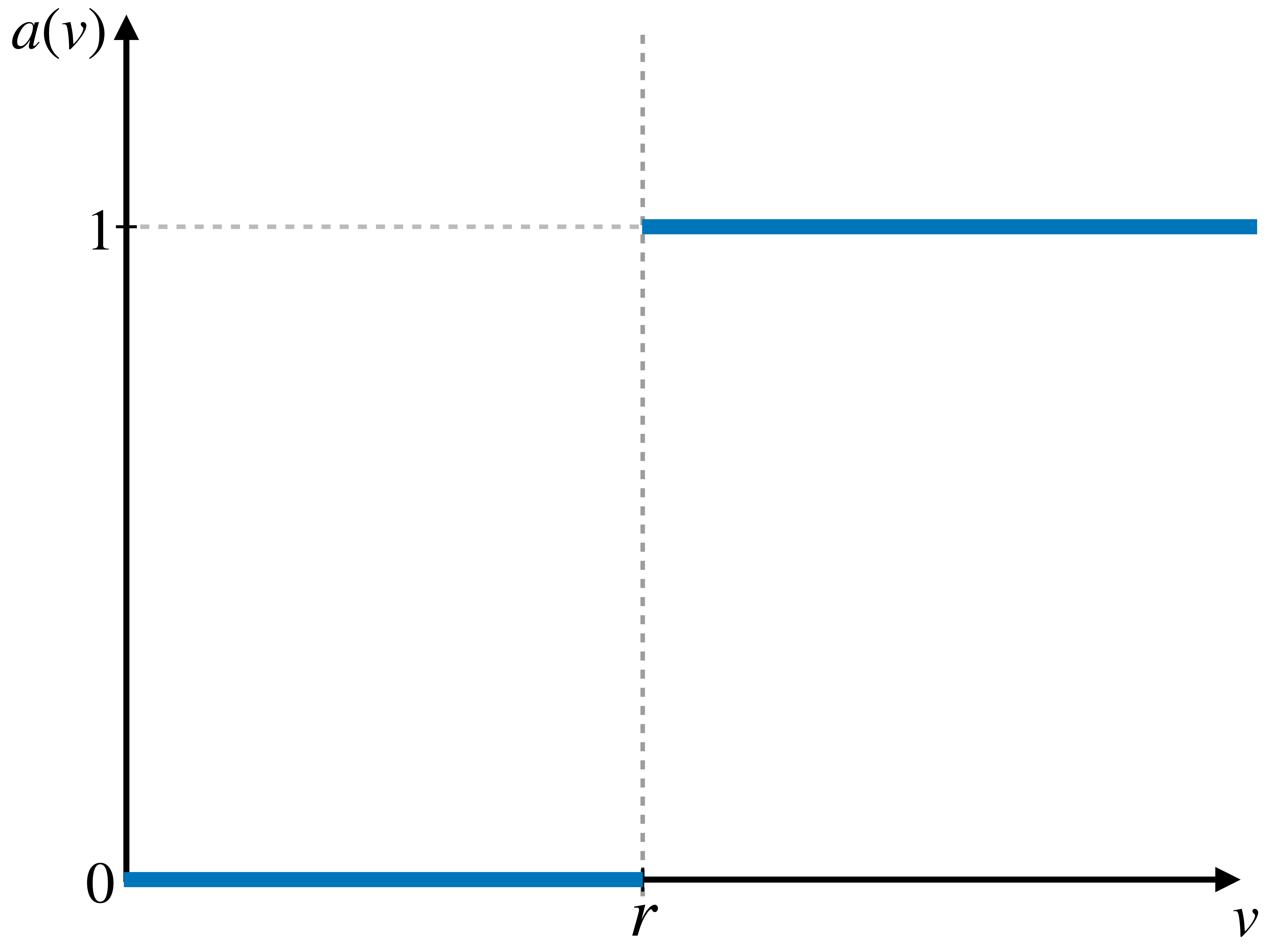

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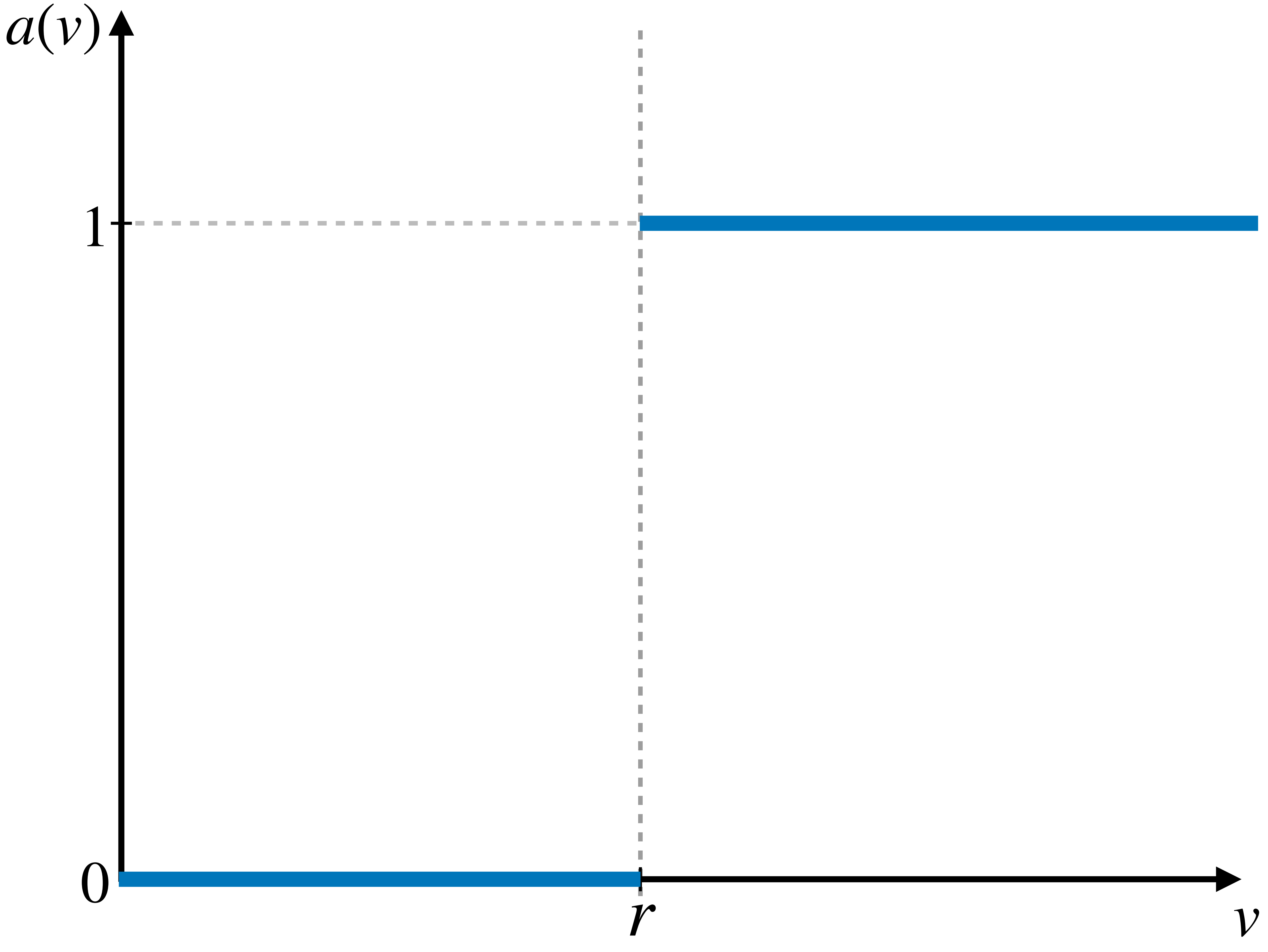

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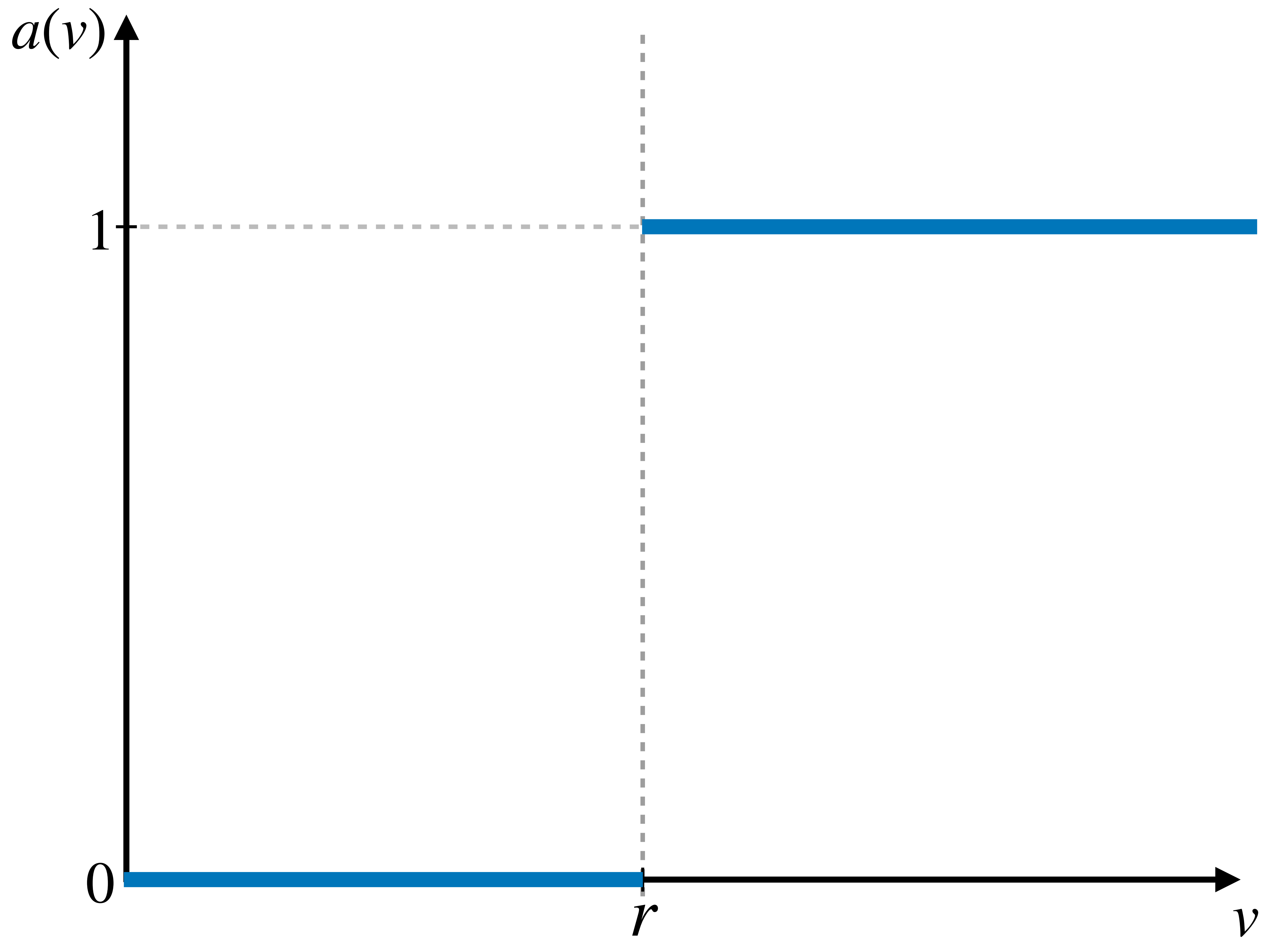

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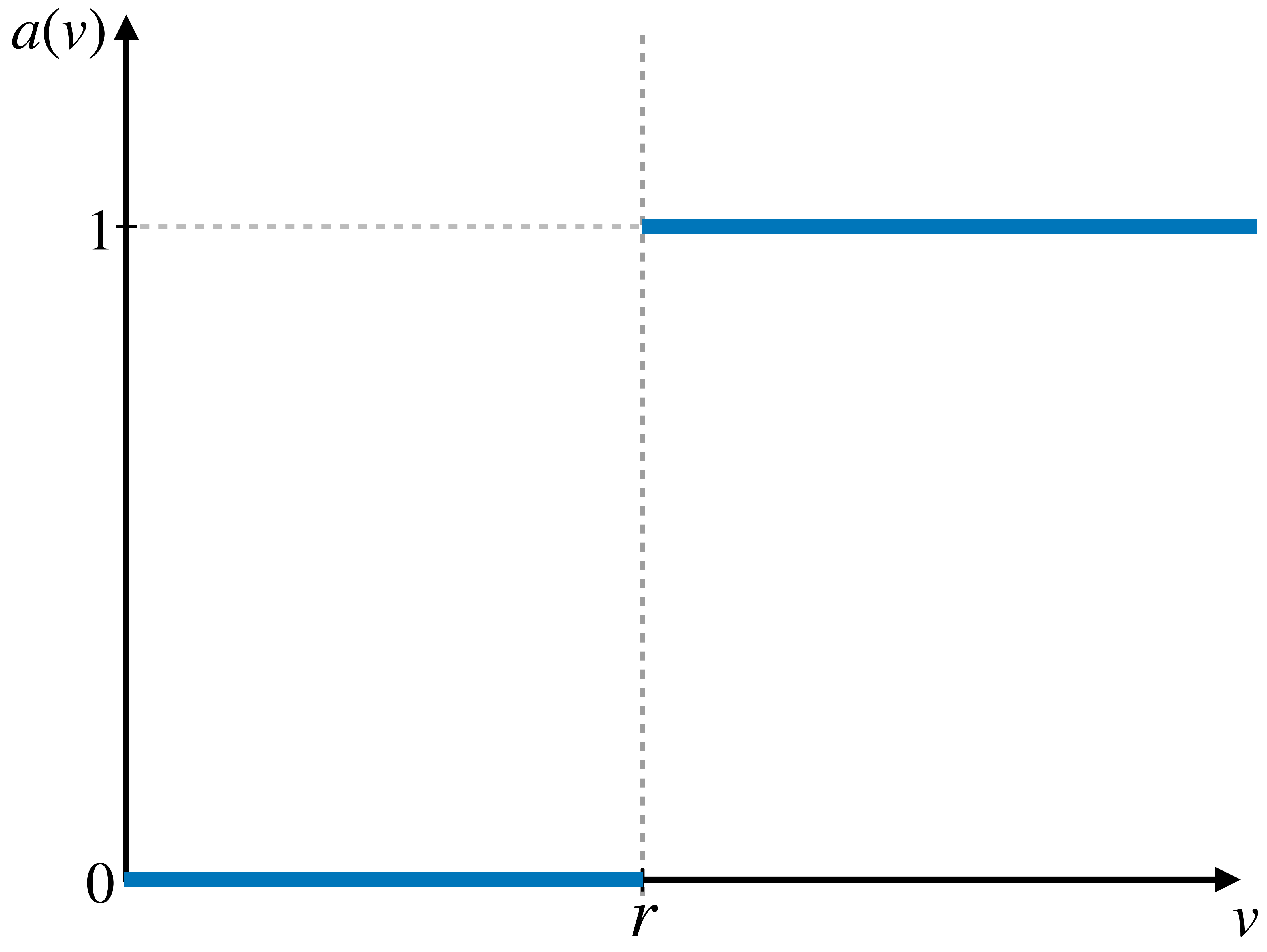

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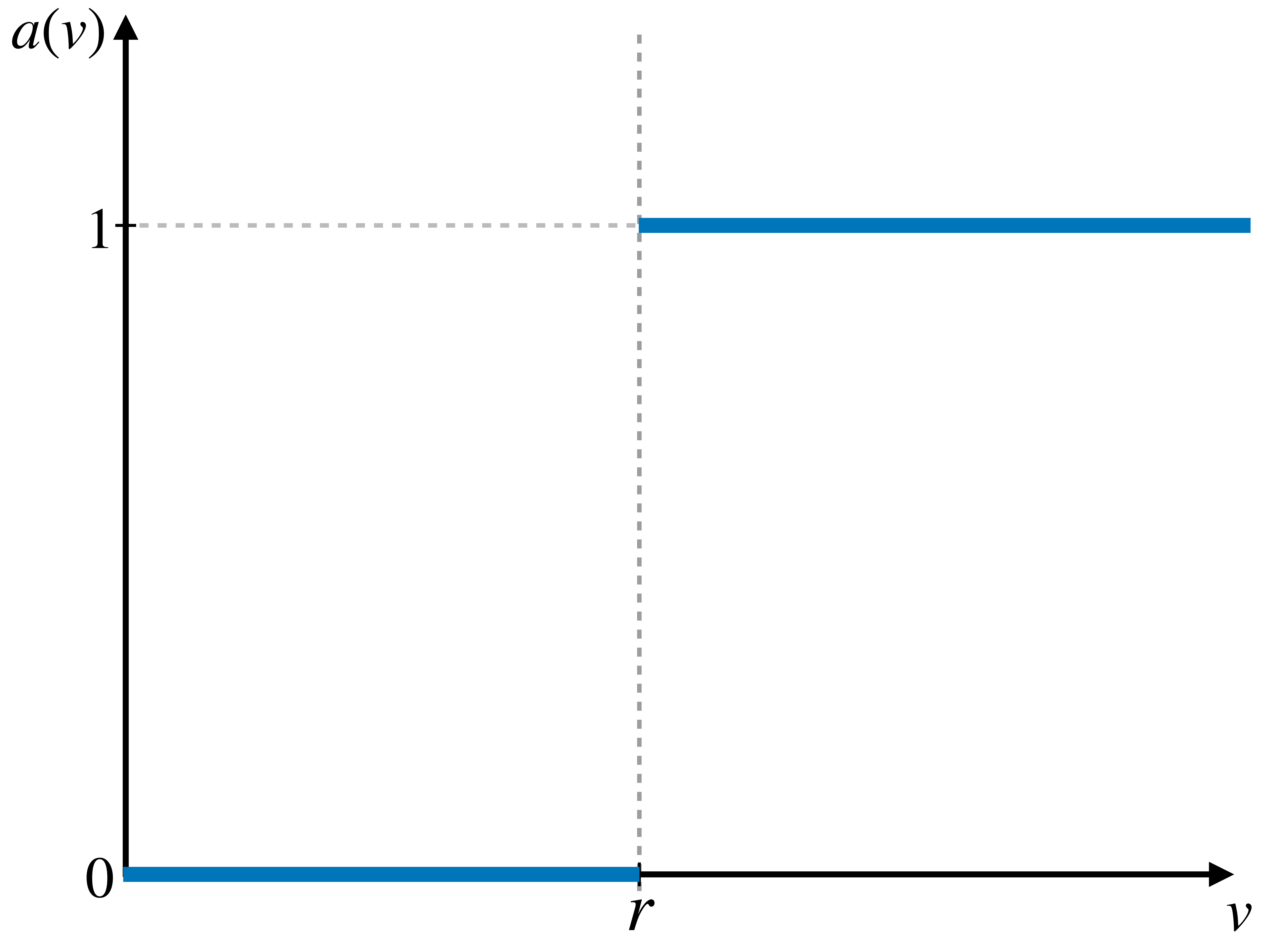

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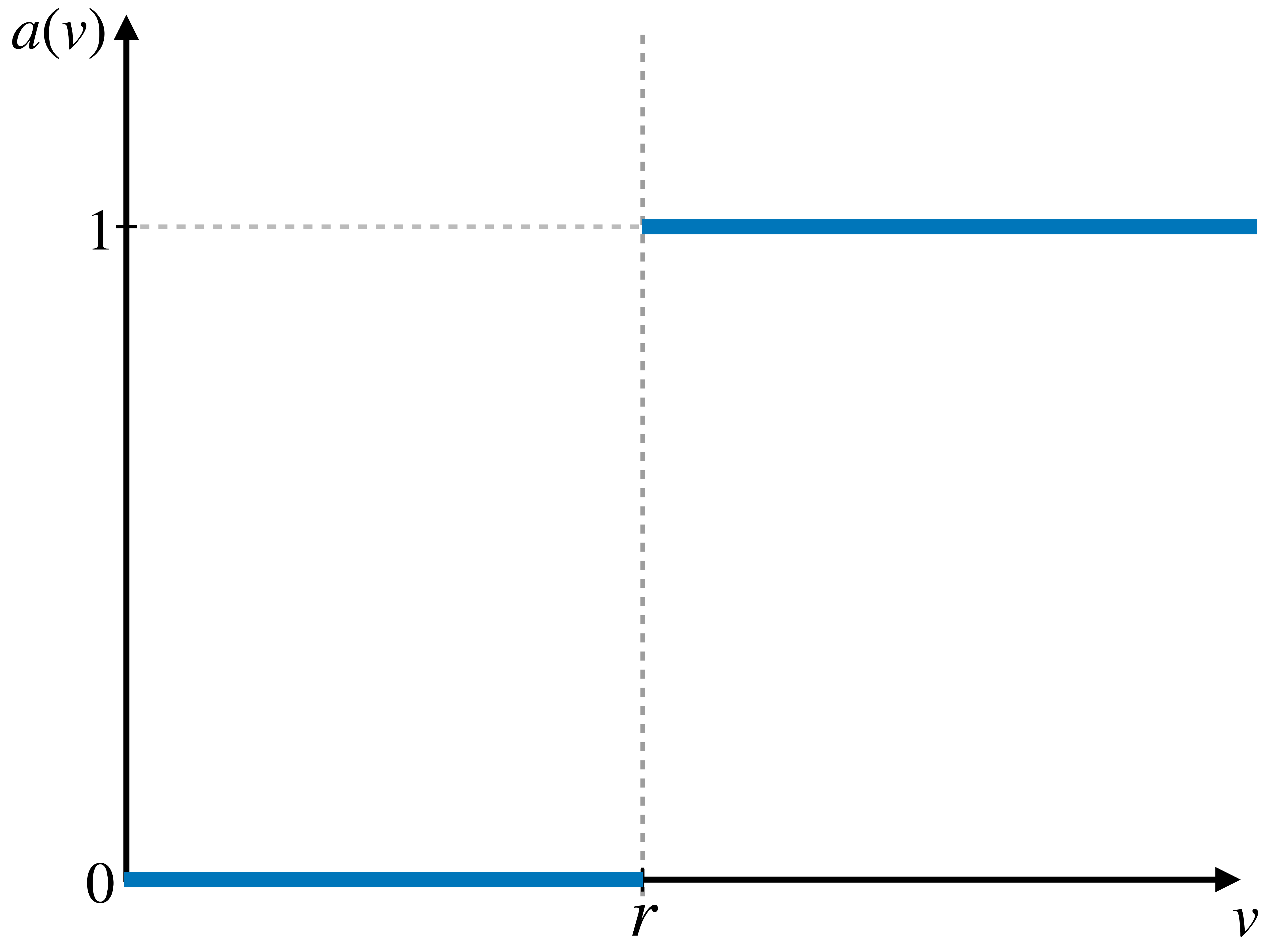

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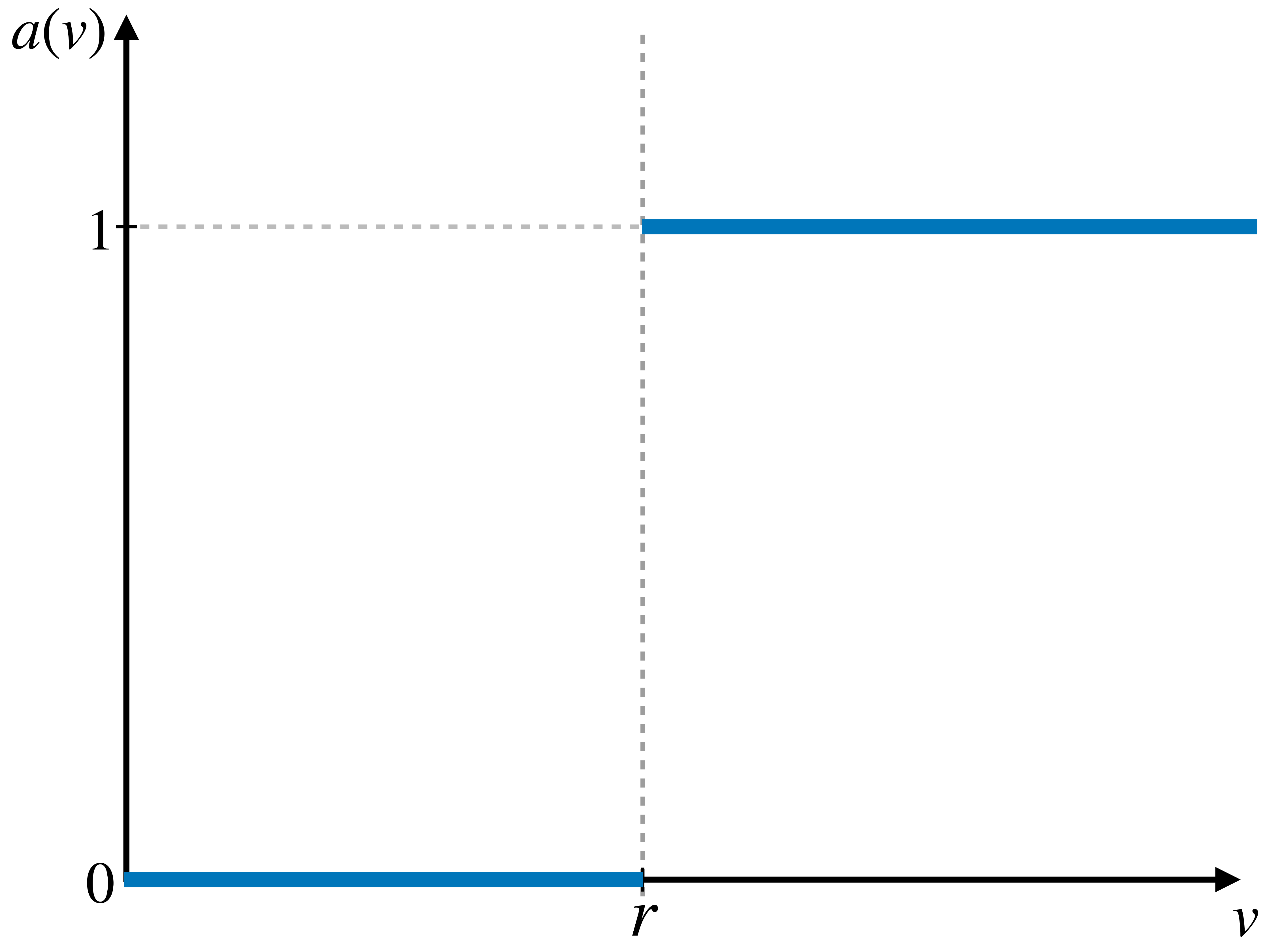

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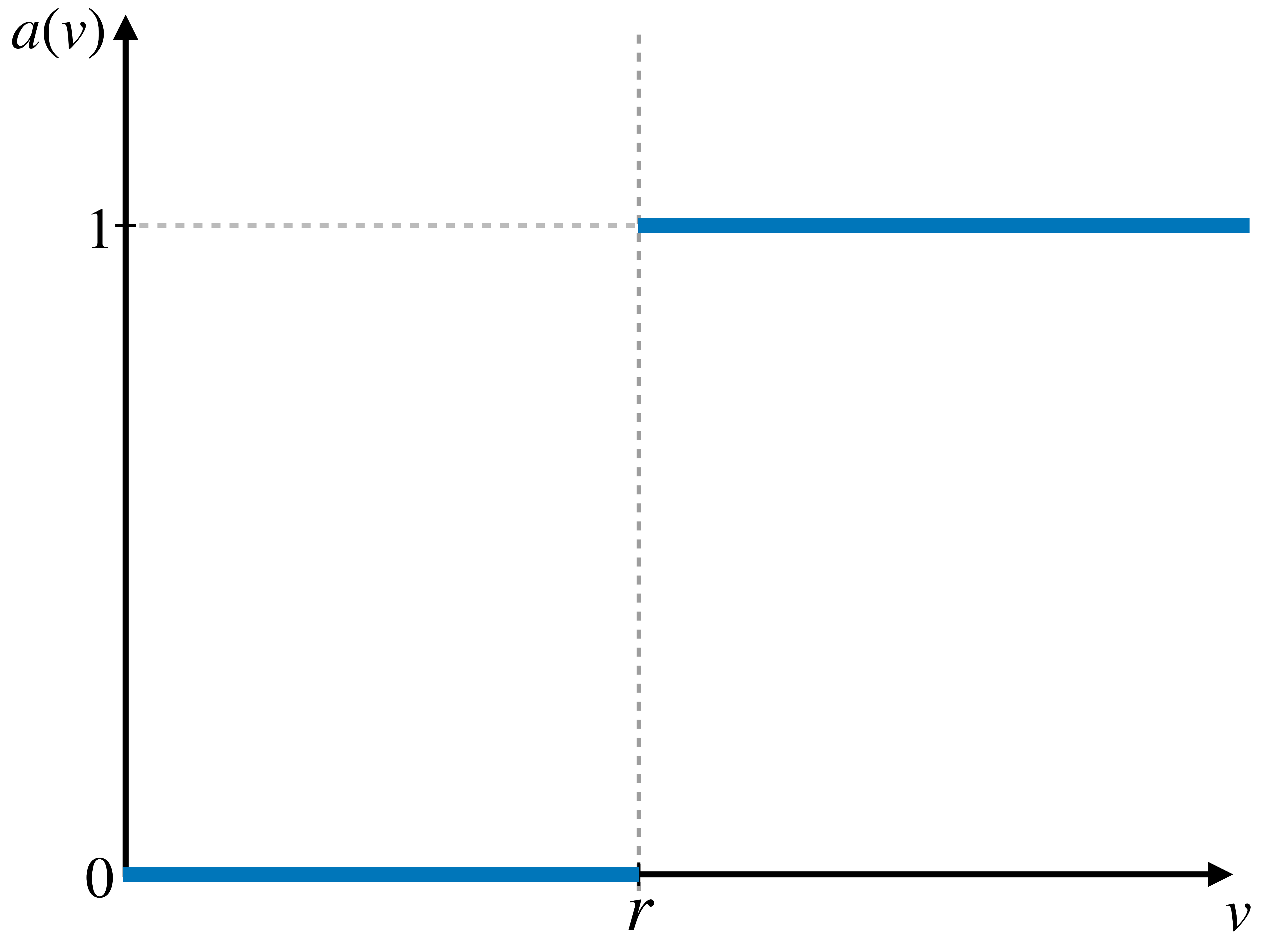

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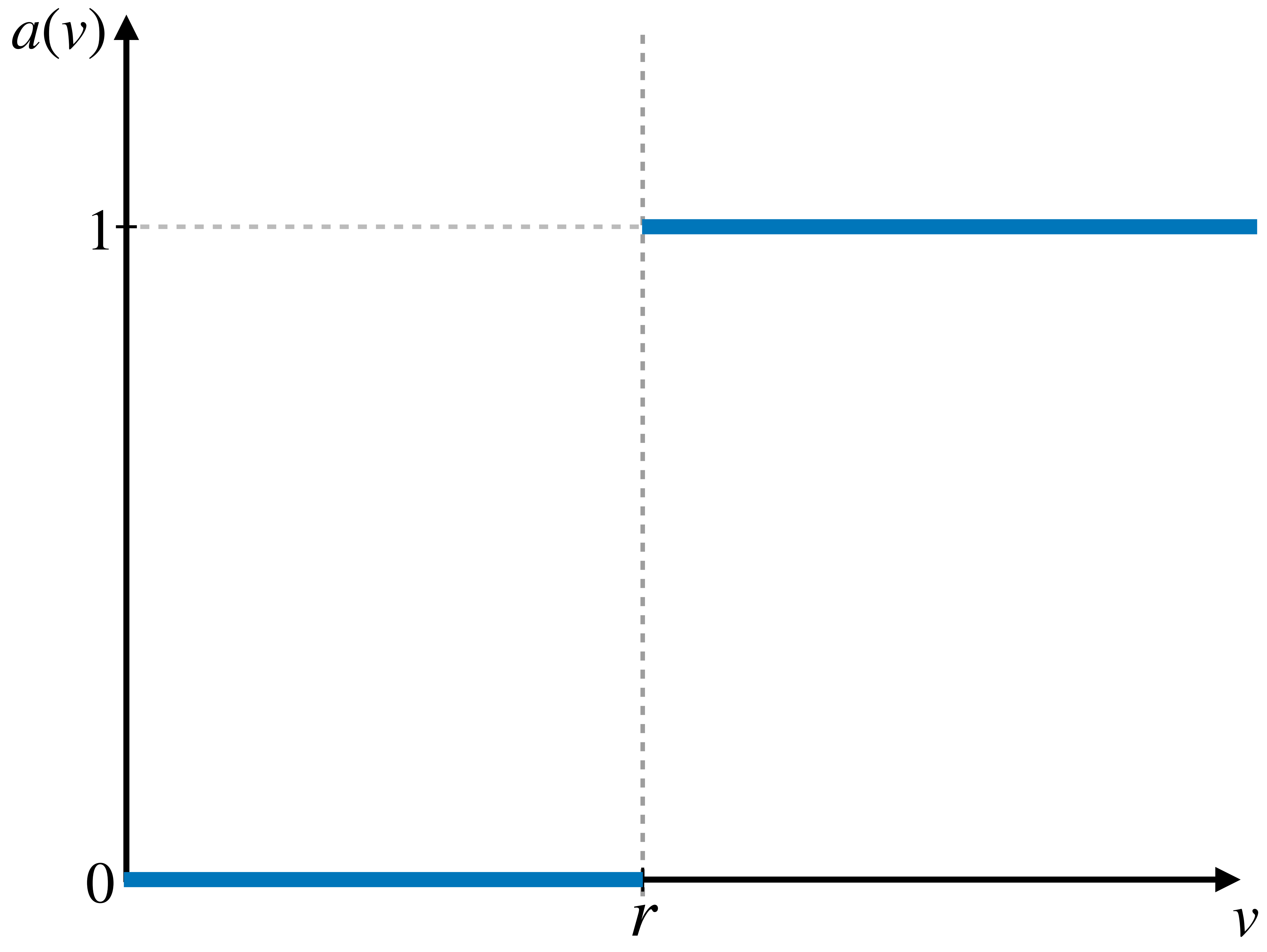

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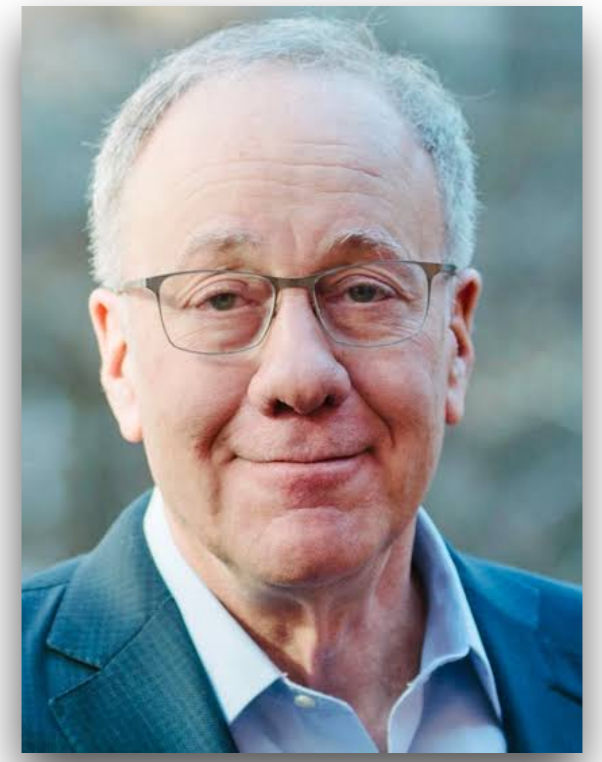

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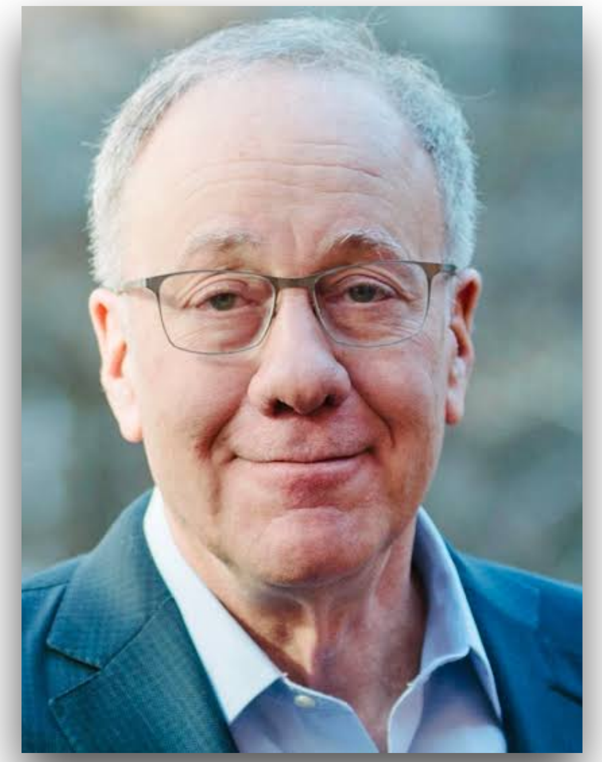
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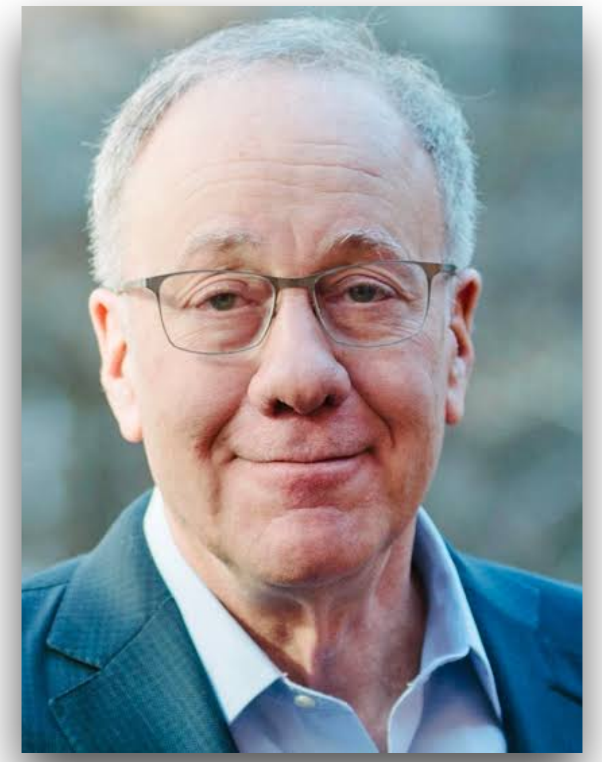

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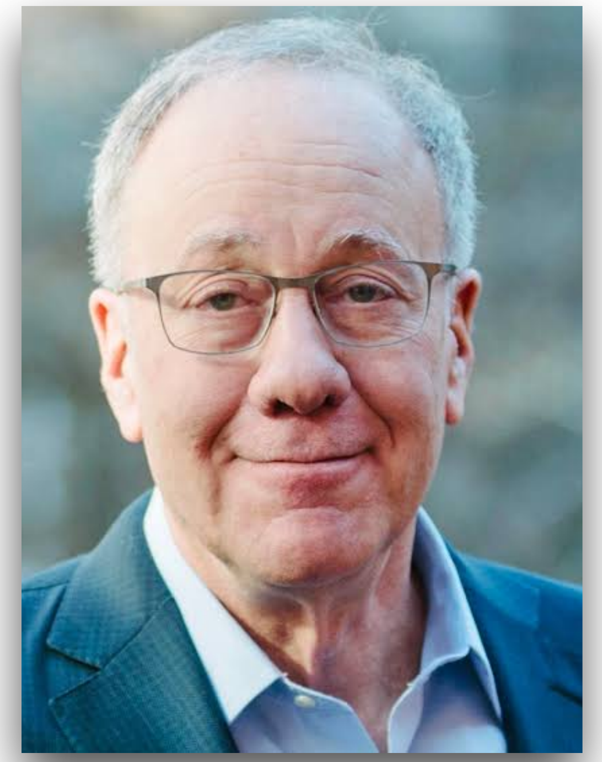

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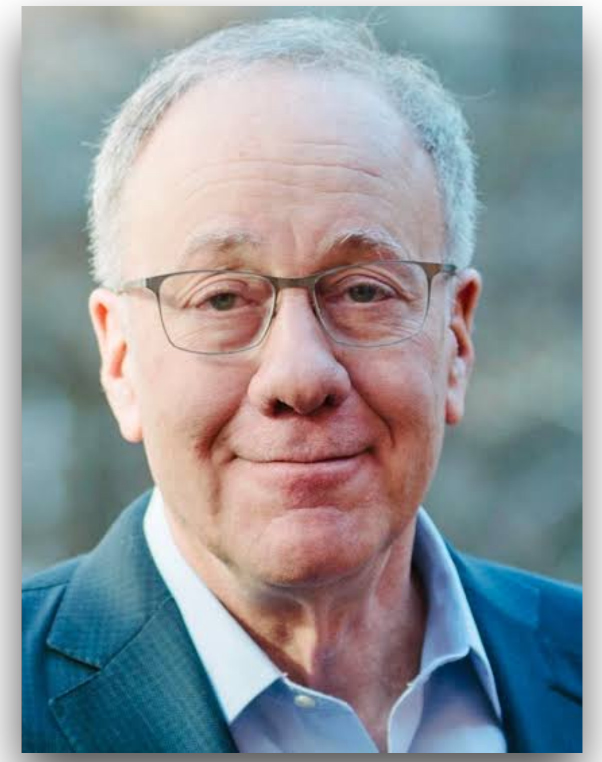
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revenue

"virtual" social welfare

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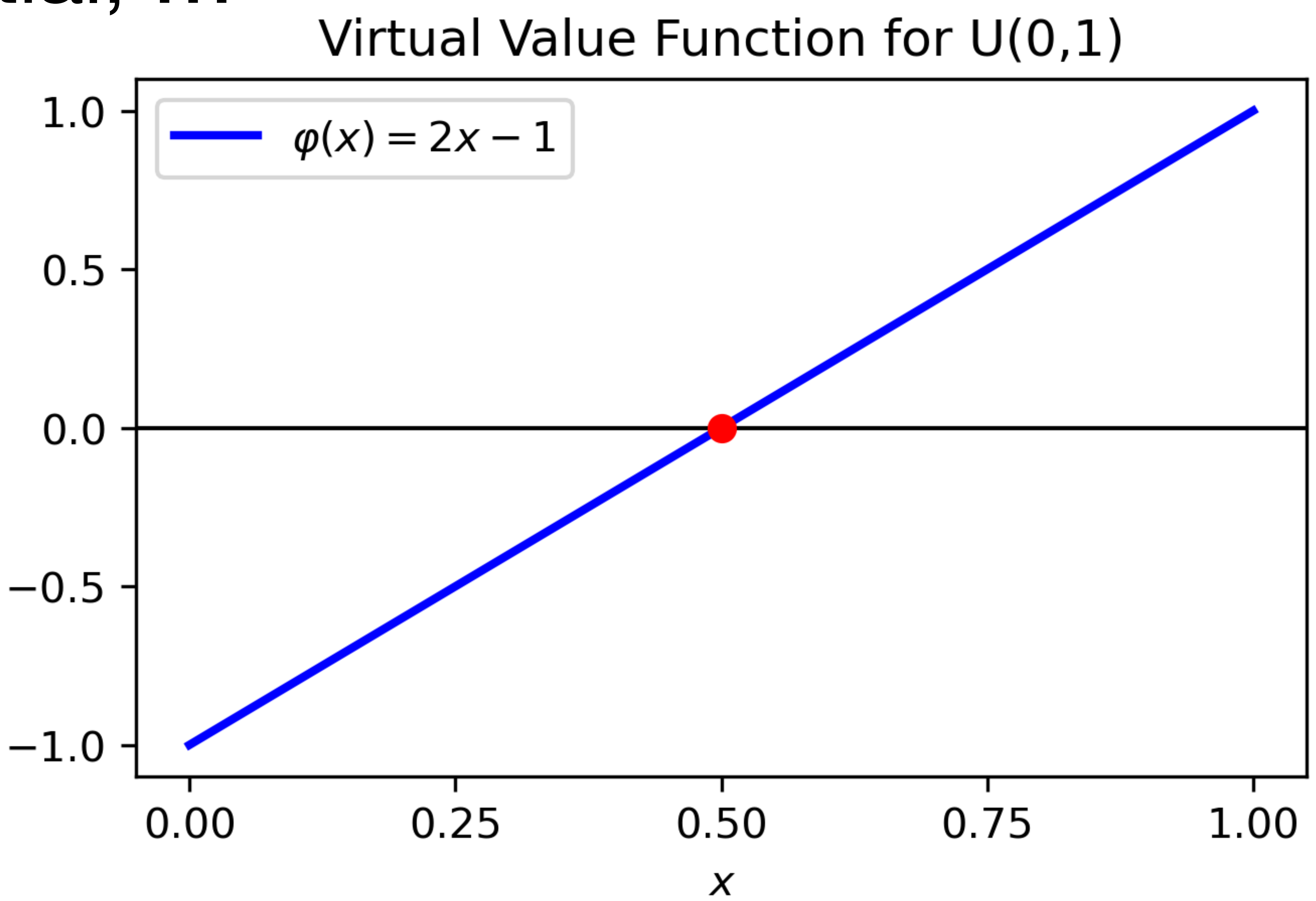

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- So, from now on let's fix a bidder $i \in [n]$ and values $\boldsymbol{v}_{-i} \in [0,1]^{n-1}$.

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For any *truthful* auction $(\boldsymbol{a}, \boldsymbol{p})$: $\mathbb{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\sum_{i=1}^{n} p_i(\boldsymbol{v})\right] = \mathbb{E}_{\boldsymbol{v}\sim\boldsymbol{F}}\left[\sum_{i=1}^{n} a_i(\boldsymbol{v})\phi_i(v_i)\right]$.

PROOF

- It is enough to prove it "per-bidder"
  - i.e., we will show that, for *any* bidder i and *all* bid/value profiles $\boldsymbol{v}_{-i}$ of the others:

$$\mathbb{E}_{v_i\sim F_i}\left[p_i(\boldsymbol{v})\right] = \mathbb{E}_{v_i\sim F_i}\left[\phi_i(v_i)x_i(\boldsymbol{v})\right].$$

  - This is enough, due to the "linearity of expectation".

- So, from now on let's fix a bidder $i \in [n]$ and values $\boldsymbol{v}_{-i} \in [0,1]^{n-1}$.

- Simplify notation: $v := v_i$, $a(v) := a_i(v, \boldsymbol{v}_{-i})$, $p(v) := p_i(v, \boldsymbol{v}_{-i})$, $F(v) := F_i(v)$, ...

$$\mathbb{E}_{v_i \sim F_i}\left[p_i(\boldsymbol{v})\right] = \mathbb{E}_{v_i \sim F_i}\left[\phi_i(v_i)x_i(\boldsymbol{v})\right]$$

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$\mathbb{E}_{v \sim F}[p(v)]$

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Interchanging Sums & Integrals

Quick Mathematical Detour

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$$\sum_{i=1}^{n} \sum_{j=1}^{i} a_{i,j}$$

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$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

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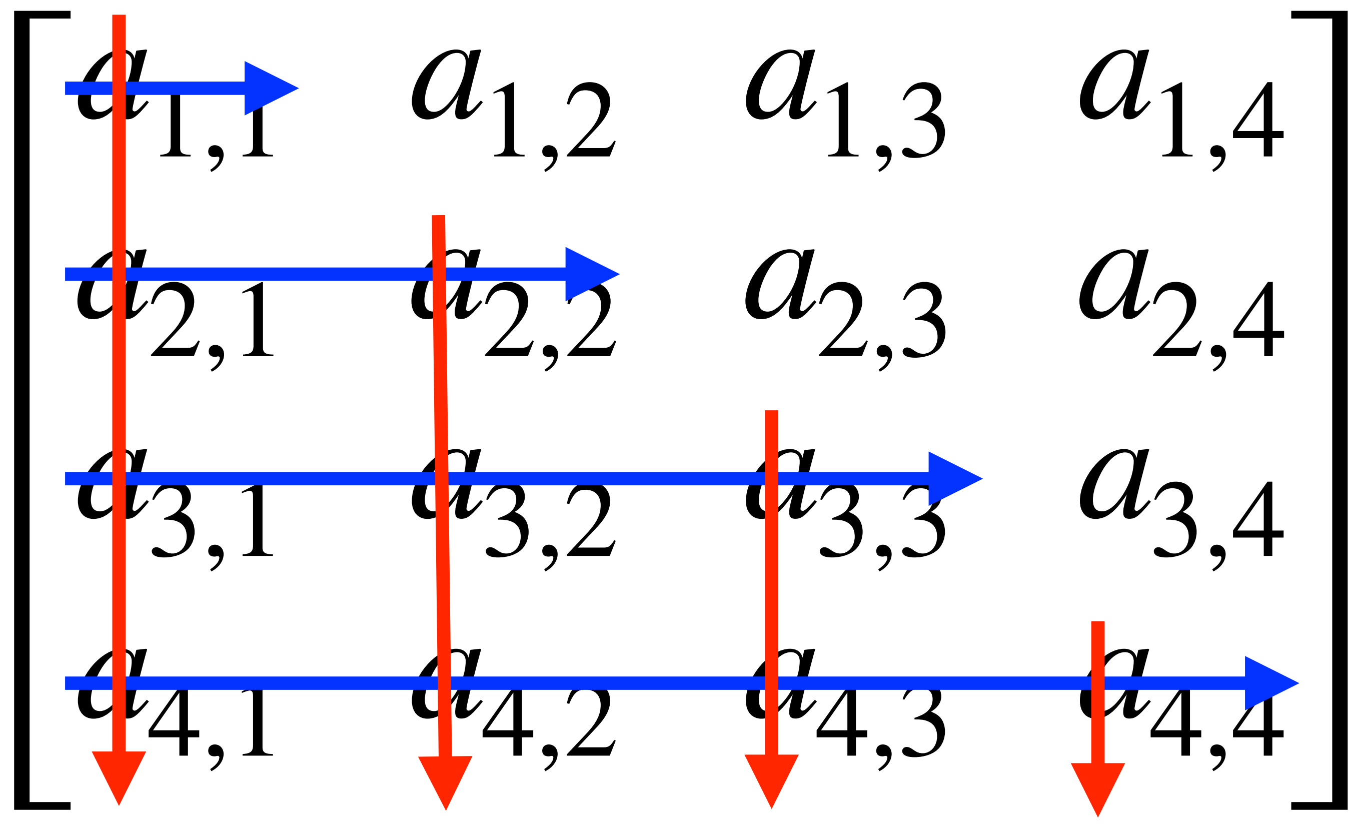

Interchanging Sums & Integrals

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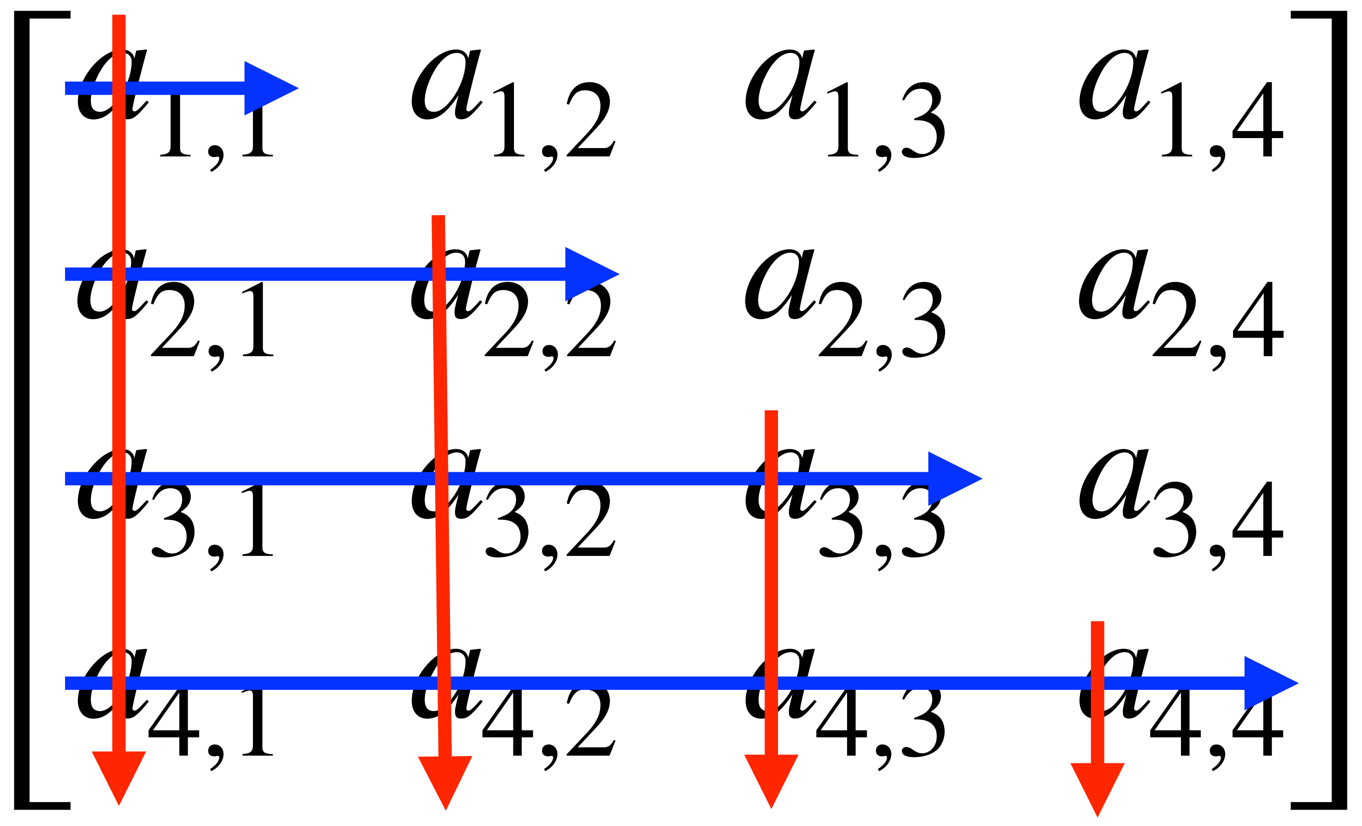
$$\sum_{i=1}^{n} \sum_{j=1}^{i} a_{i,j} = \sum_{j=1}^{n} \sum_{i=j}^{n} a_{i,j}$$



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\end{aligned}$$

“Simplicity” vs Optimality

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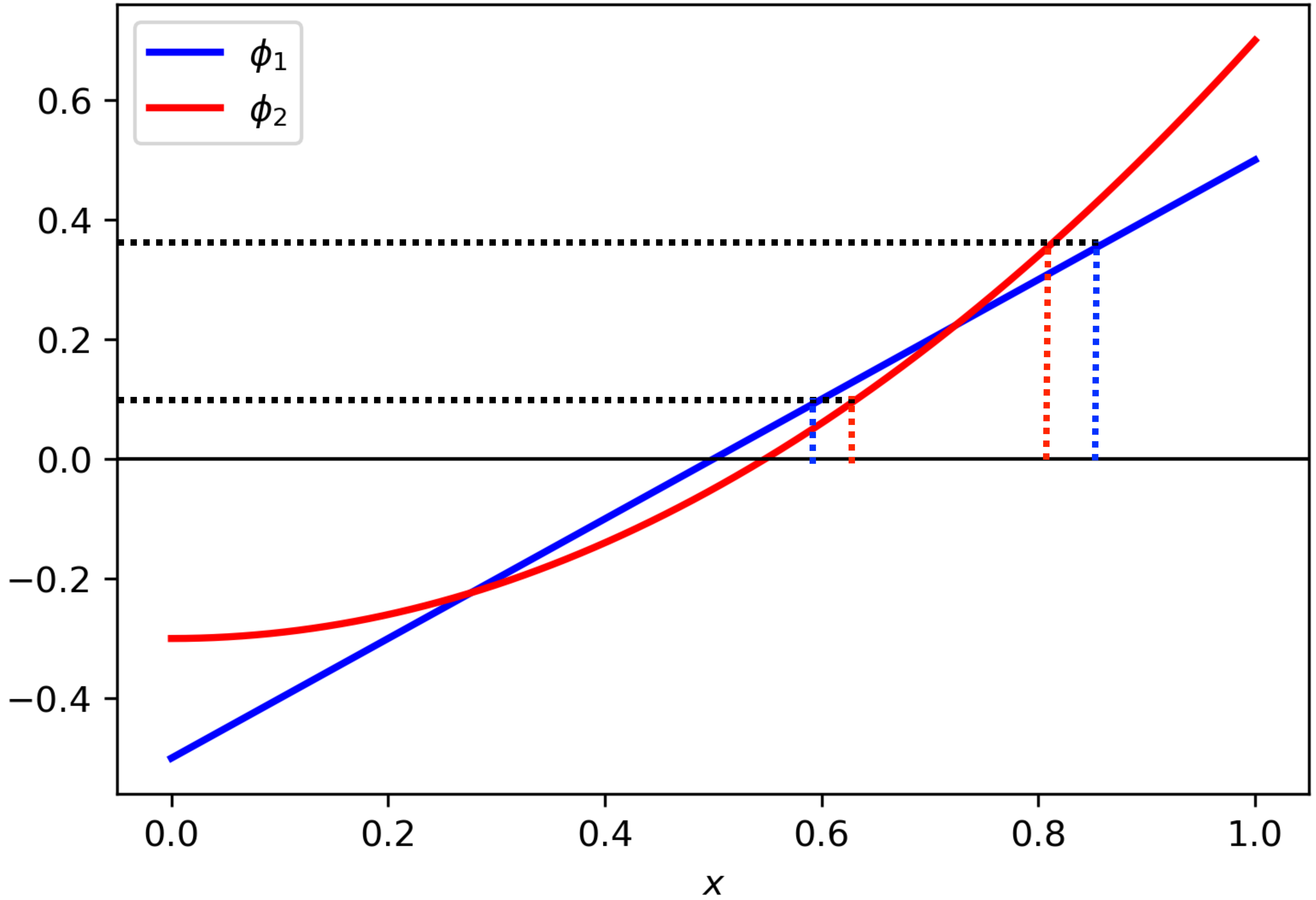

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The Bullow-Klemperer Approximation

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THEOREM (J. Bulow & P. Klemperer [1996])

For regular iid priors, the expected revenue of the second-price auction (with no reserve) on $n + 1$ bidders is at least the expected revenue of the optimal auction with on n bidders.

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COROLLARY

For n bidders with regular iid priors, the second-price auction achieves at least a $\frac{n-1}{n}$-fraction of the optimal expected revenue.

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Optimal Revenue Approximation for Non-Identical Regular Bidders

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$$\frac{1}{2} = 0.5$$

A Small Glimpse Beyond: Multi-item Auctions

Multi-Dimensional Revenue Maximization

Complications

Multi-Dimensional Revenue Maximization

Complications

- Fundamental technical obstacles, even for a *single* bidder!
- Randomization is required, in general, for optimality
  - Uncountably infinitely many “menus”, even for two items.
- Computational hardness barriers
- Large constant approximations only (e.g., 8)
- <u>Generally</u>: the exact structure, and key computational properties, of the optimal auctions **still elude us**!
  - Resolved only for a single-bidder, small number of items, and very specific distributions (most notably, uniform)

Single-Bidder, Uniform IID

How do optimal auctions look like?

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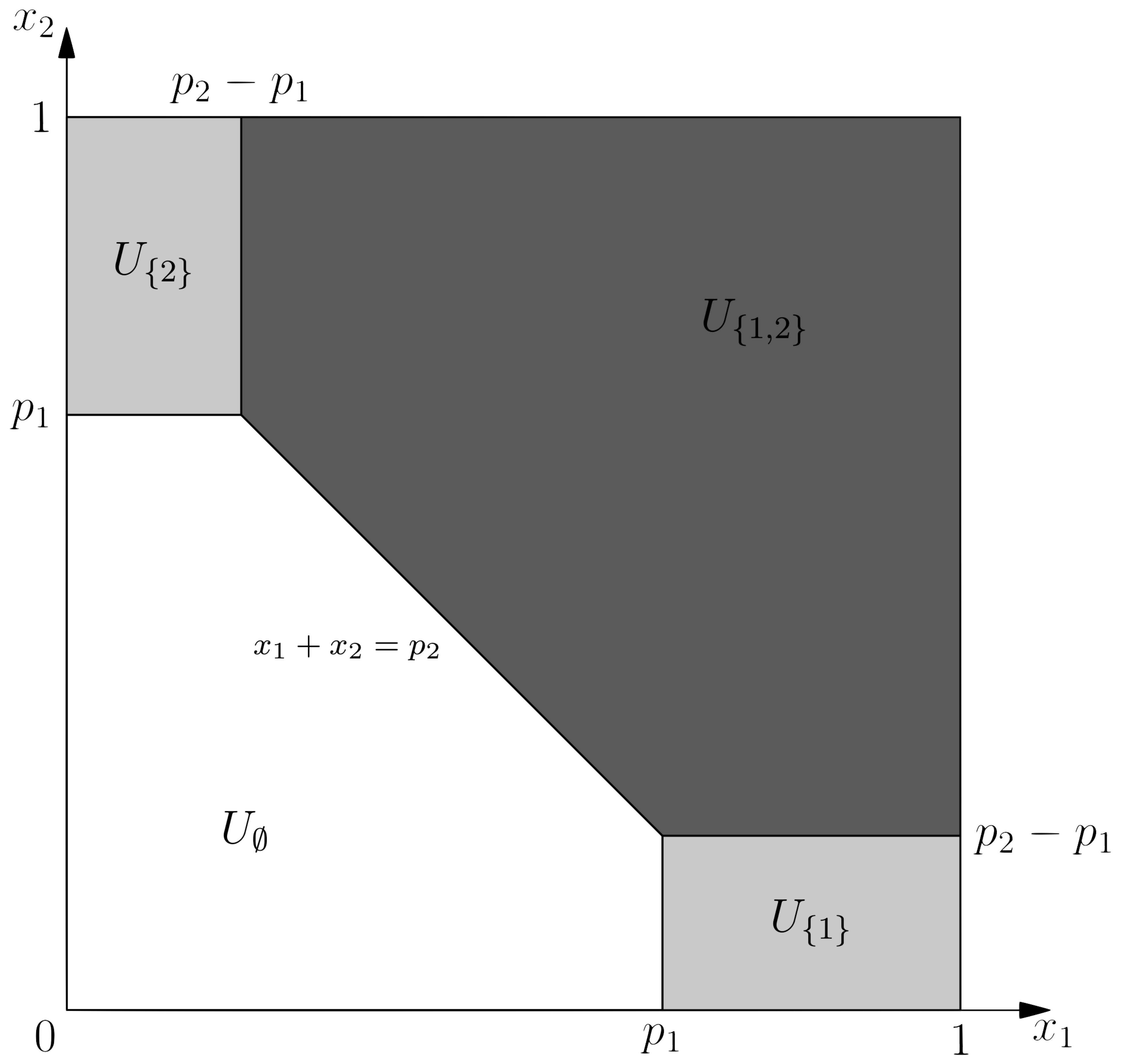

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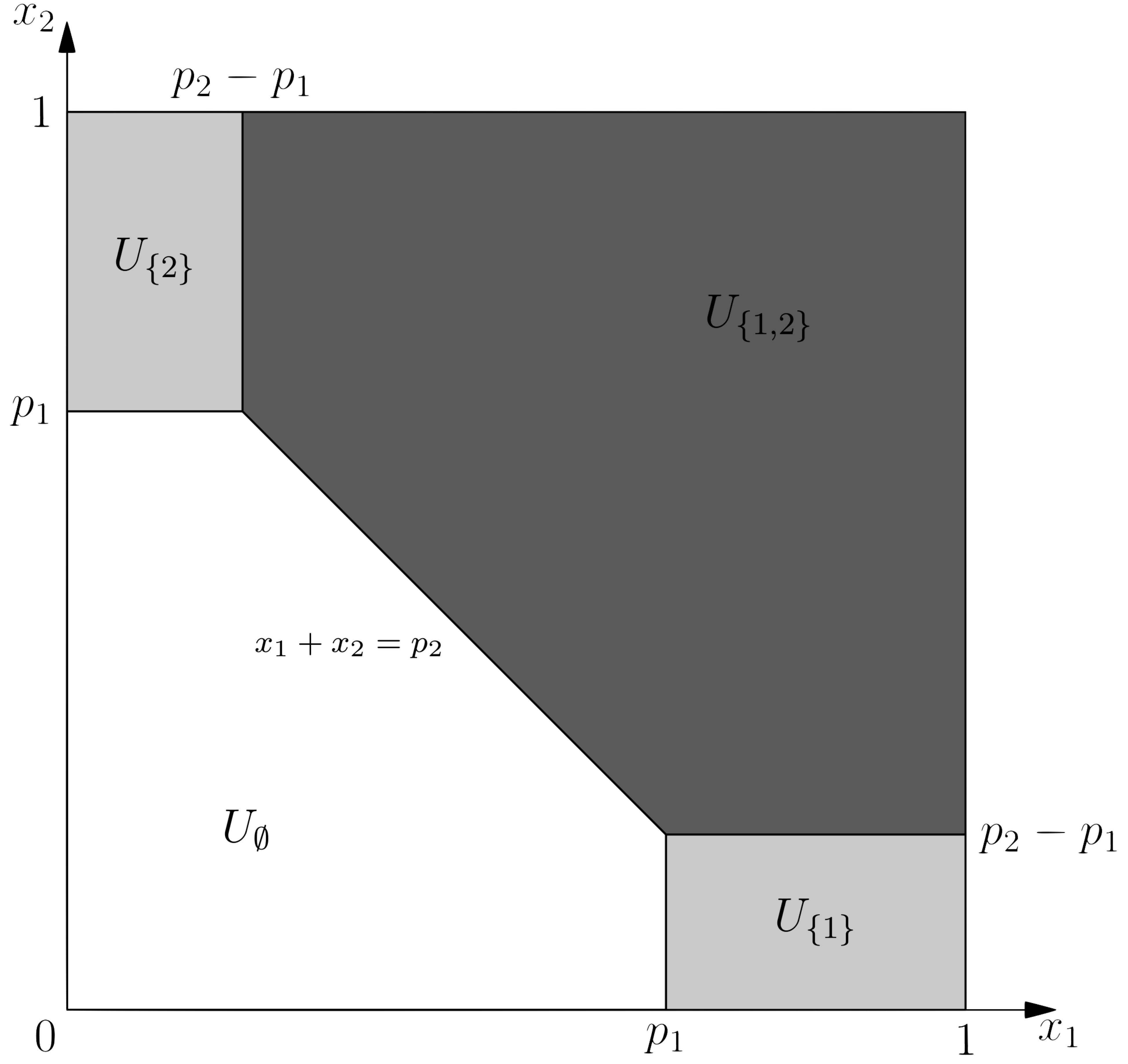

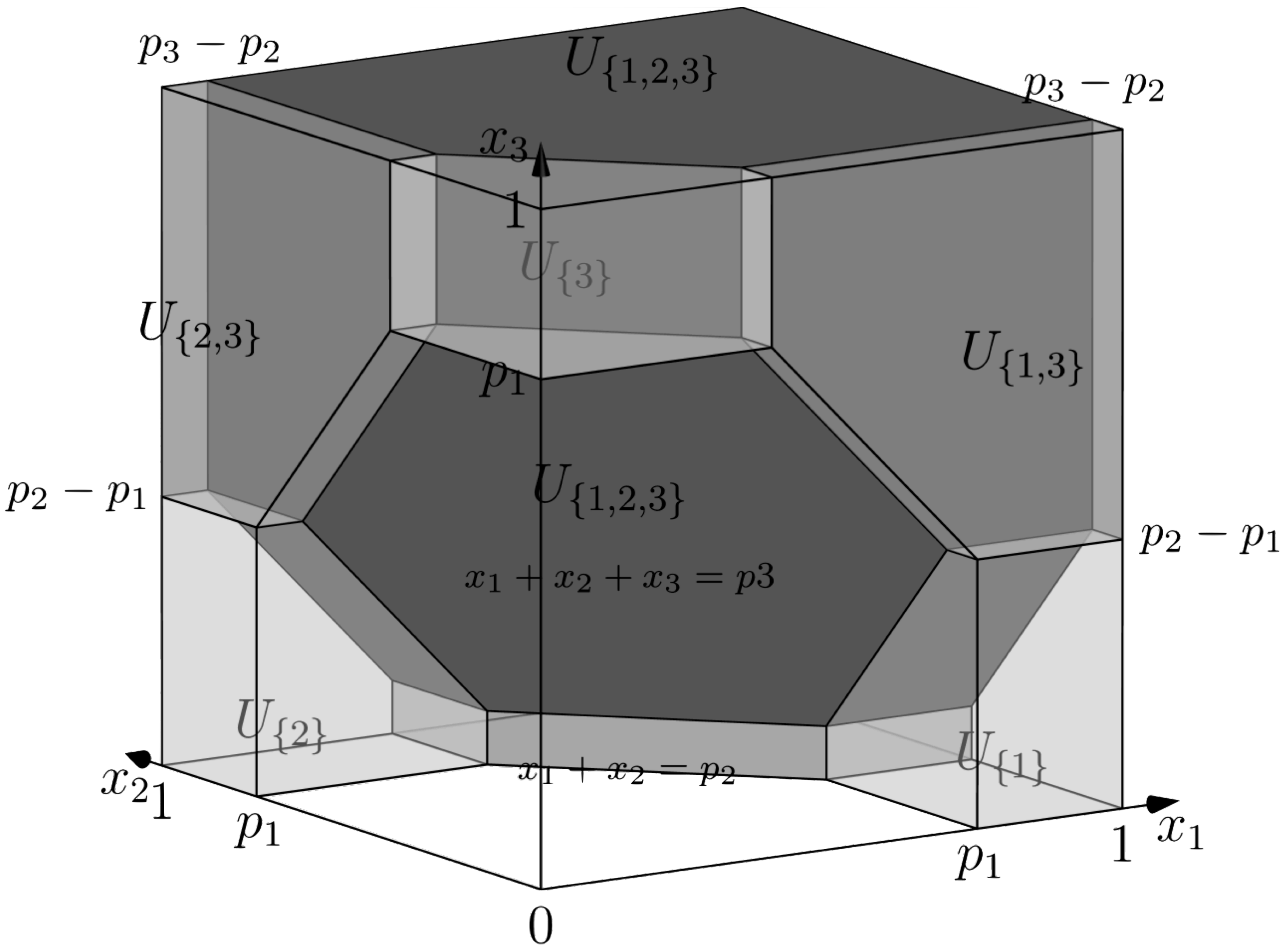

The “Straight-Jacket” Auction

A colouring puzzle

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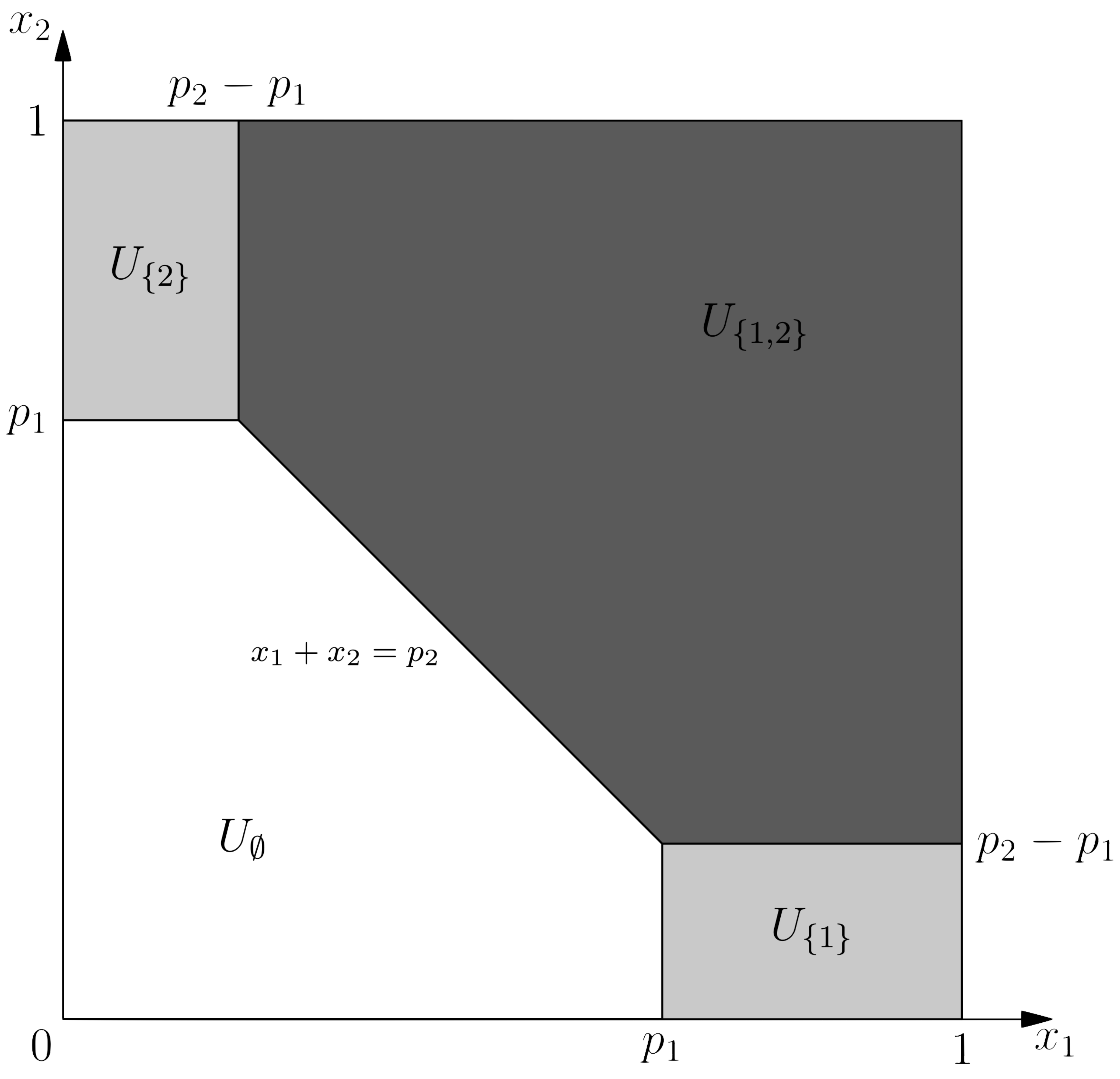

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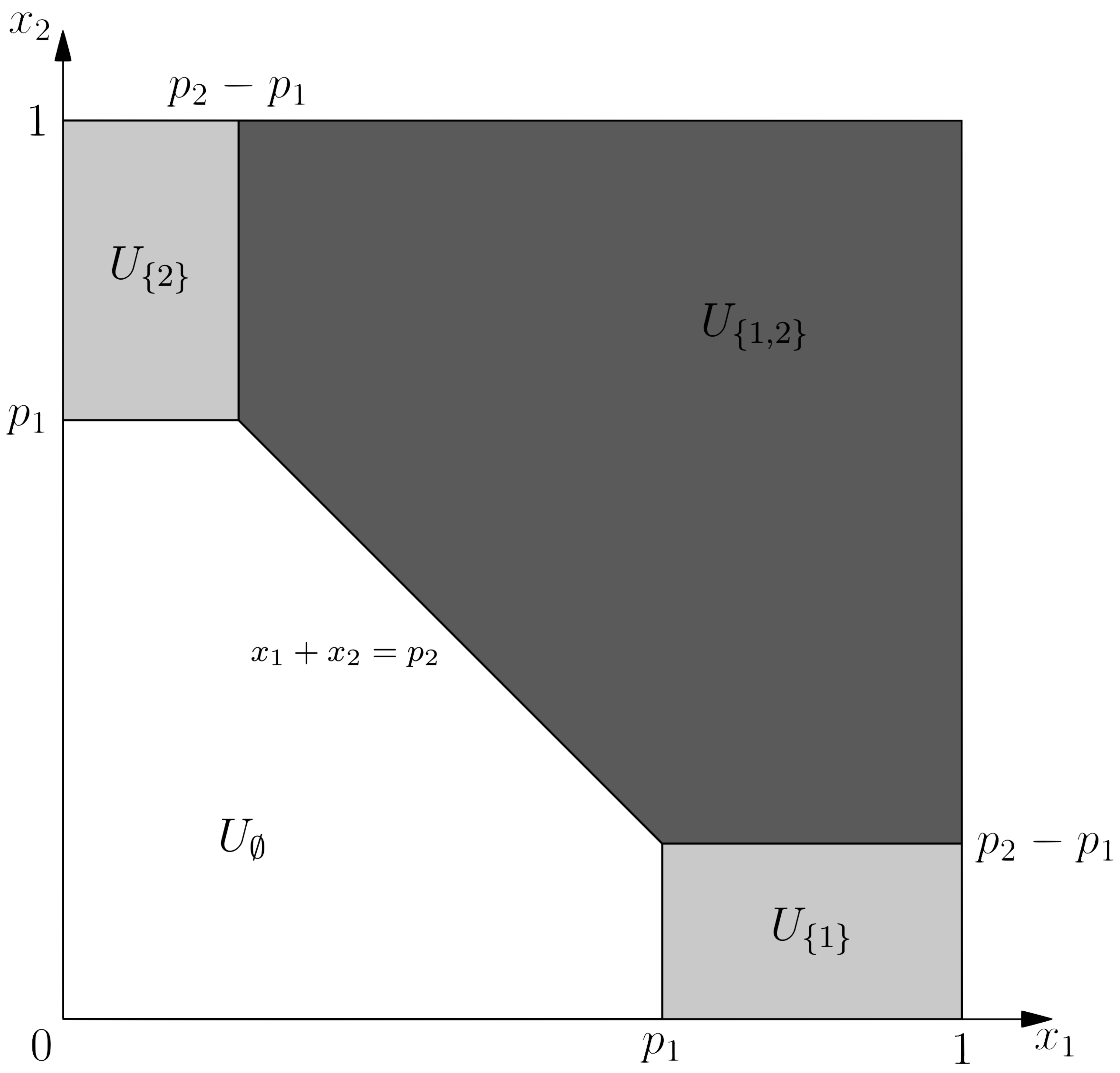

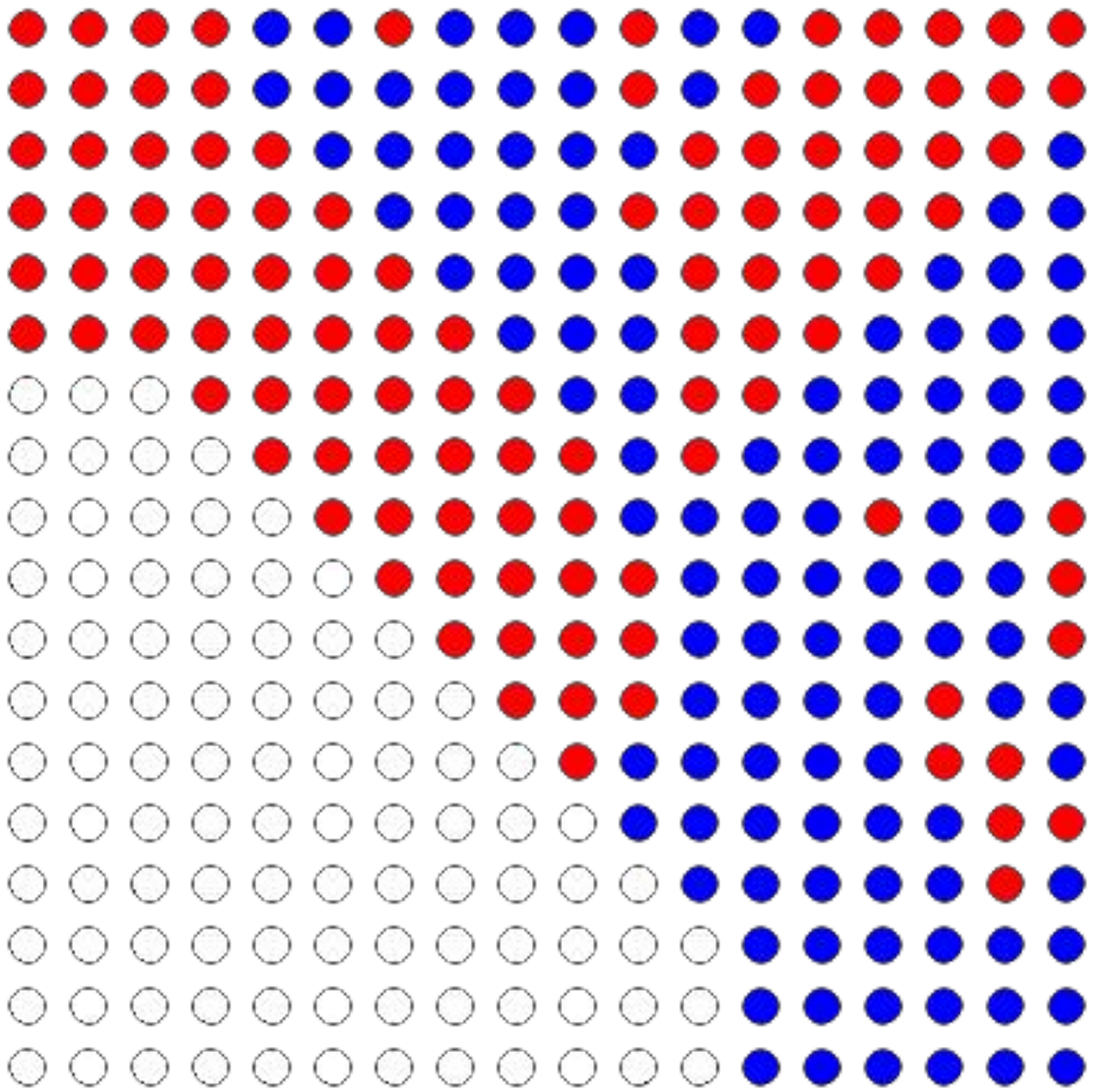

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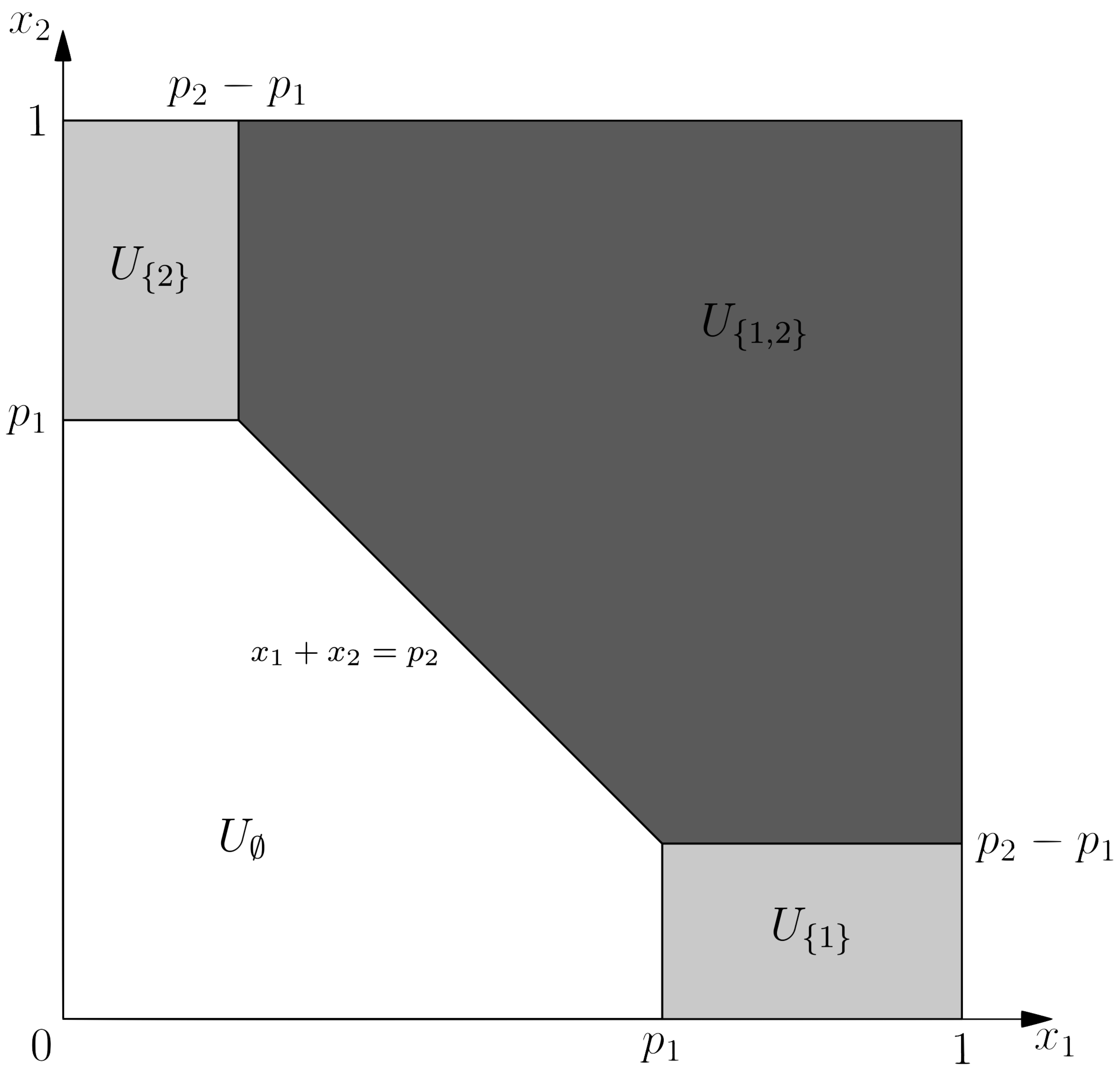

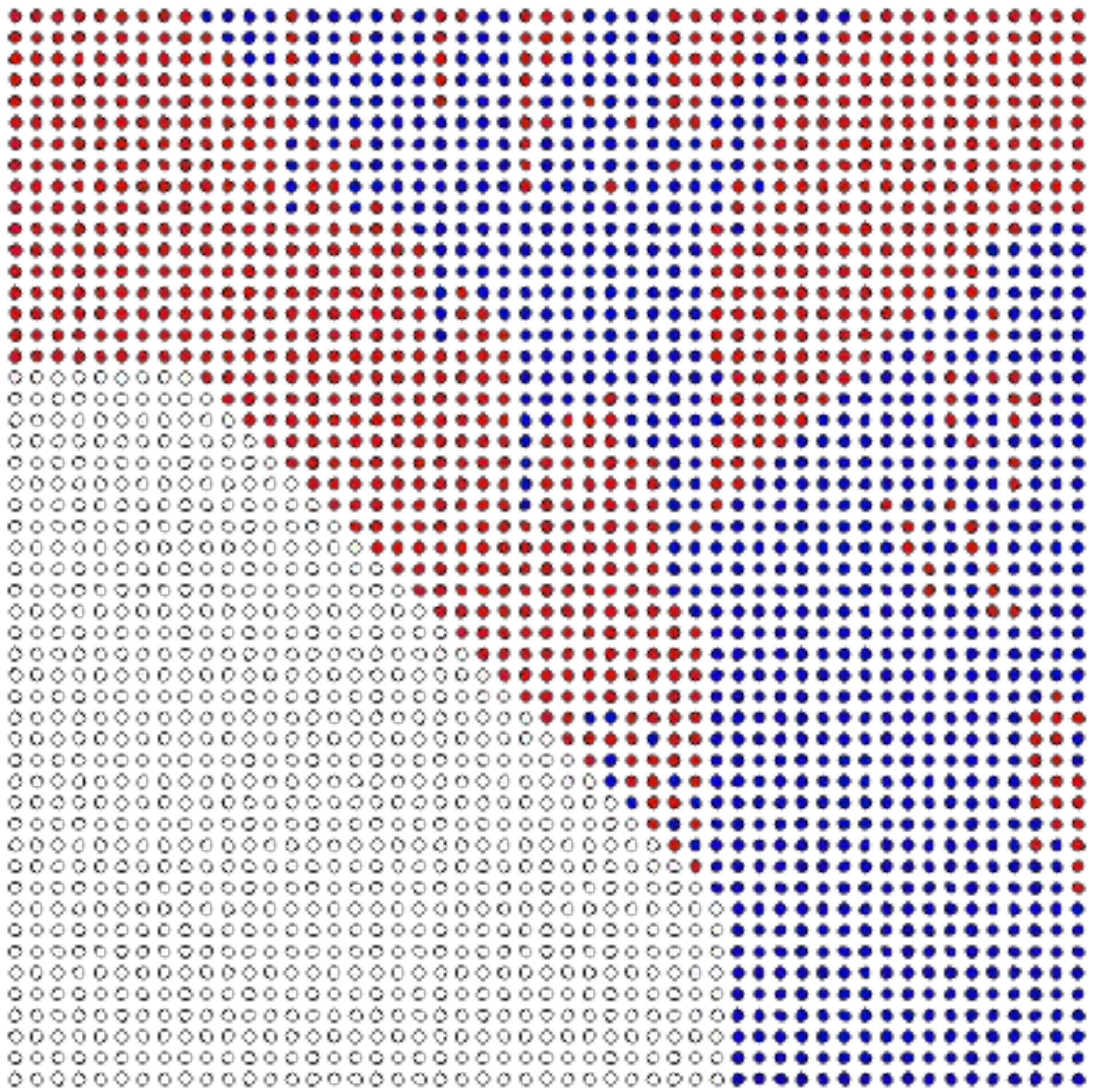

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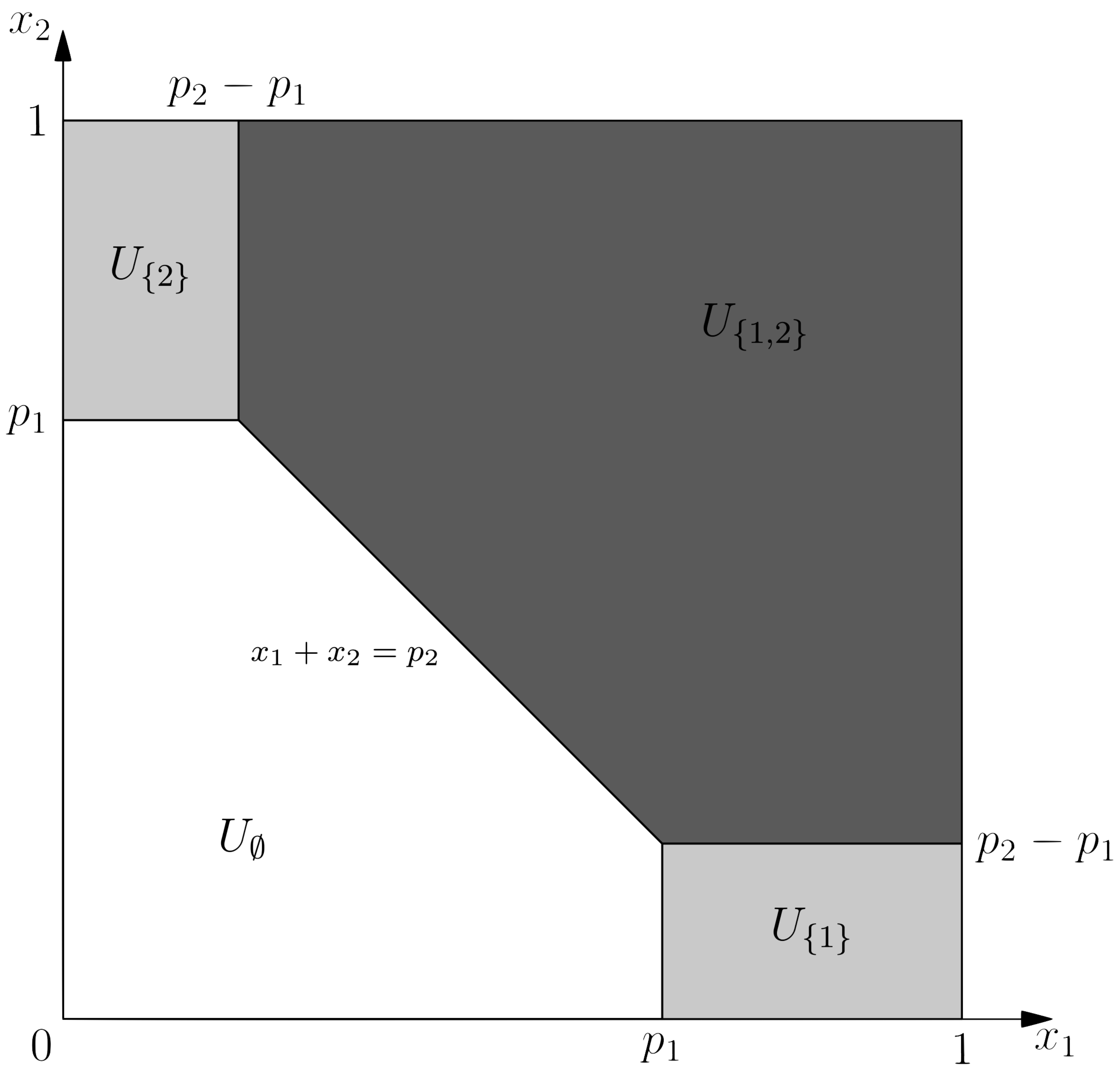

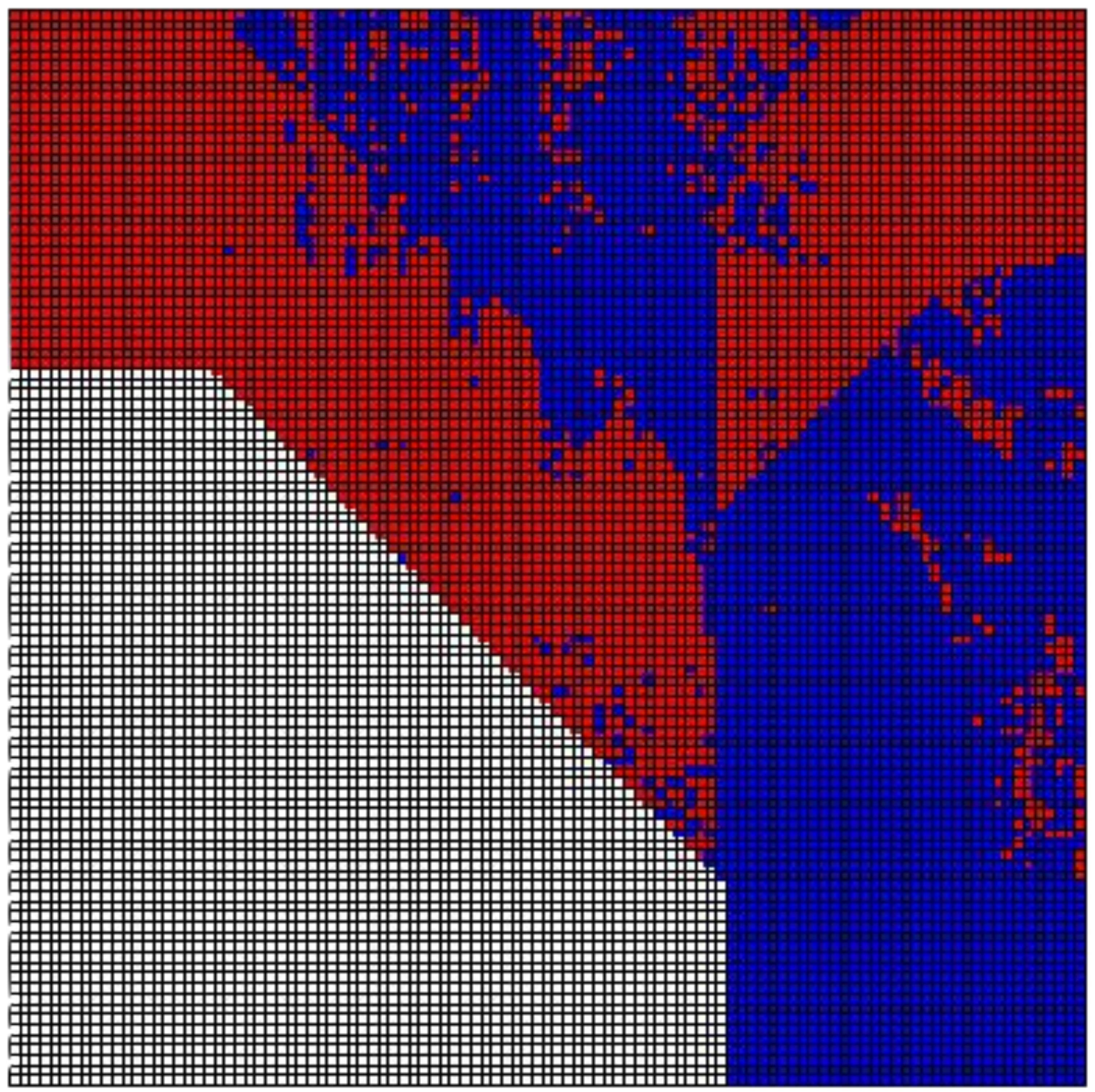

Thank you!