Algorithmic Game Theory and Applications

A proof of Nash's Theorem

A Feature of Best Responses

Claim: The best response is either *pure*, or there are *infinitely many* best responses.

Proof:

Assume that we have a best response strategy x_i which is not pure.

That means that the support of x_i contains at least two pure strategies s_i^1 and s_i^2.

Each of those pure strategies, if played as pure strategies, should give the same utility to the player (by Proposition 2).

And this utility is the maximum the player can get with a best response.

Any convex combination (probability mixture) of those two yields maximum utility, i.e., it is a best response.

There are infinitely many convex combinations of those two pure strategies.

Nash Equilibrium Existence

Nash's Theorem (1950): Every finite, normal-form game has at least one mixed Nash equilibrium.

Next, we will see a proof (sketch) of that theorem.

We will consider different levels to the proof.

Level 1: We will prove the theorem using a theorem from topology (Brouwer's fixed point theorem) as a tool.

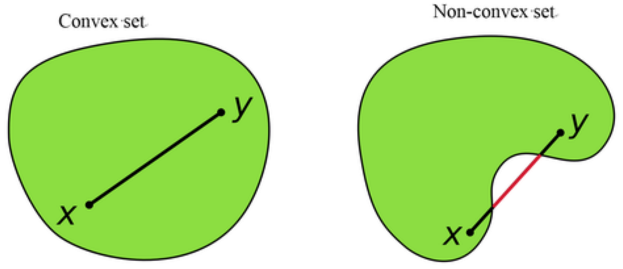
Level 2: We will prove the theorem from topology (Brouwer's fixed point theorem) using a different lemma from topology (Sperner's Lemma).

Level 3: We will prove the lemma from topology (Sperner's Lemma) from first principles.

The exposition follows Shoham and Leyton-Brown, Ch. 3.3.4

Technical Definitions

Convexity: A set $C \subset \mathbb{R}^m$ is convex if for every $x, y \in C$ and $\lambda \in [0,1]$, we have that $\lambda x + (1 - \lambda)y \in C$.



Source: Wikipedia

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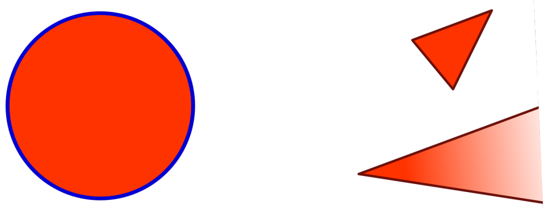
Convexity: A set $C \subset \mathbb{R}^m$ is convex if for every $x, y \in C$ and $\lambda \in [0,1]$, we have that $\lambda x + (1-\lambda)y \in C$.

Compactness: A set $C \subset \mathbb{R}^m$ is compact if it is closed and bounded.

Closed: contains its boundary (its limit points).

Bounded: there is a bounded distance between every two points.

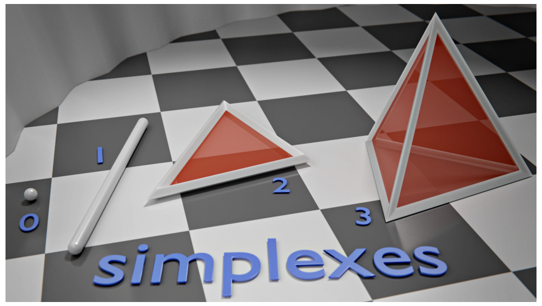


Source: Wikipedia

Example of a convex compact set

Recall the unit simplex:

$$\left\{ y \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} y_i = 1, \forall i = 1,\ldots,n, y_i \geq 0 \right\}$$



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Nash Equilibrium Existence

Brouwer's Fixed Point Theorem (1911): Let $C \subset \mathbb{R}^m$ be convex and compact, and let $f : C \to C$ be a continuous function. Then f has a fixed point, i.e., there exists some point $x \in C$ such that $f(x) = x$.

Intuition: We would like our function to map mixed strategy profiles to mixed strategy profiles, and the fixed point to correspond to the mixed Nash equilibrium of our game.

Question: What will be our convex, compact set?

The set of all mixed strategy profiles

i.e., $\Delta(S_1) \times \ldots, \times \Delta(S_n)$

Nash Equilibrium Existence

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We need to define our function:

We can define it separately for each component i, i.e., we can define $f_i : C \rightarrow \Delta(S_i)$.

Ideas?

Maybe define f_i to be the best response of player i?

Not a continuous function!

Nash Equilibrium Existence

Increase in utility by playing s_i, capped below by 0.

continuous

Let $f_{i,s_i}(x) = \max\{0,\ u_i(s_i, x_{-i}) - u_i(x)\}$

convex and compact (simplotope)

Define $f: \Delta(S_1) \times \ldots, \times \Delta(S_n) \to \Delta(S_1) \times \ldots, \times \Delta(S_n)$ by $f(x) = x'$ where

$$x_i'(s_i) = \frac{x_i(s_i) + f_{i,s_i}(x)}{\sum_{b_i \in S_i}\left(x_i(b_i) + f_{i,b_i}(x)\right)} = \frac{x_i(s_i) + f_{i,s_i}(x)}{1 + \sum_{b_i \in S_i} f_{i,b_i}(x)}$$

continuous

By Brouwer's fixed point theorem, f has a fixed point, where $f(x) = x$.

It remains to show that x is a mixed Nash equilibrium.

Nash Equilibrium Existence

$$x_i'(s_i) = \frac{x_i(s_i) + f_{i,s_i}x)}{\sum_{b_i \in S_i}\left(x_i(b_i) + f_{i,b_i}(x)\right)} = \frac{x_i(s_i) + f_{i,s_i}(x)}{1 + \sum_{b_i \in S_i} f_{i,b_i}(x)}$$

thus,

$$x_i'(s_i) \cdot \left(1 + \sum_{b_i \in S_i} f_{i,b_i}(x)\right) = x_i(s_i) + f_{i,x_i}(x)$$

hence,

$$x_i(s_i) \cdot \sum_{b_i \in S_i} f_{i,b_i}(x) = f_{i,s_i}(x)$$

Here we used the fact that $f(x_i(s_i)) = x_i'(s_i) = x_i(s_i)$ which is true because x' is a fixed point.

We will show that

$\Rightarrow f_{i,b_i} = 0 \ \forall b_i \in S_i \quad \Rightarrow x_i$ is a best response

Nash Equilibrium Existence

We have $x_i(s_i) \cdot \sum_{b_i \in S_i} f_{i,b_i}(x) = f_{i,s_i}(x)$

Claim: There exists at least one pure strategy $c_i \in S_i$ in the support of x_i such that $f_{i,c_i} = 0$.

Proof: Recall $f_{i,s_i}(x) = \max\{0,\ u_i(s_i, x_{-i}) - u_i(x)\}$

Also recall that $u_i(x) = \sum_{i: s_i \in \text{ supp}(x_i)} x_i(s_i) \cdot u_i(s_i, x_{-i})$

This implies that there exists c_i such that $u_i(c_i, x_{-i}) \leq u_i(x_i) \ \Rightarrow f_{i,c_i} = 0$.

Nash Equilibrium Existence

This implies that there exists c_i such that $u_i(c_i, x_{-i}) \leq u_i(x_i) \Rightarrow f_{i,c_i} = 0$.

From the previous slide, we have

$x_i(s_i) \cdot \sum_{b_i \in S_i} f_{i,b_i}(x) = f_{i,s_i}(x)$ for all $s_i \in S_i$

Also in particular for c_i, for which $x_i(c_i) \cdot \sum_{b_i \in S_i} f_{i,b_i}(x) = 0$

It cannot be the case that $x_i(c_i) = 0$ (why?)

This means that $\sum_{b_i \in S_i} f_{i,b_i}(x) = 0$, but we know that $f_{i,b_i} \geq 0$ for all i by definition.

This can only mean one thing: $f_{i,b_i} = 0$ for all i.

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Let $f_{i,s_i}(x) = \max\{0,\ u_i(s_i, x_{-i}) - u_i(x)\}$

Define $f : \Delta(S_1) \times \ldots, \times \Delta(S_n) \to \Delta(S_1) \times \ldots, \times \Delta(S_n)$ by $f(x) = x'$ where

$$x_i'(s_i) = \frac{x_i(s_i) + f_{i,s_i}(x)}{\sum_{b_i \in S_i} \left(x_i(b_i) + f_{i,b_i}(x)\right)} = \frac{x_i(s_i) + f_{i,s_i}(x)}{1 + \sum_{b_i \in S_i} f_{i,b_i}(x)}$$

At a fixed point of f, we have $f_{i,s_i} = 0$ for all $i \Rightarrow \sigma$ is a mixed Nash equilibrium.

Nash Equilibrium Existence

Nash's Theorem (1950): Every finite, normal-form game has at least one mixed Nash equilibrium.

Next, we will see a proof (sketch) of that theorem.

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Level 1: We will prove the theorem using a theorem from topology (Brouwer's fixed point theorem) as a tool.

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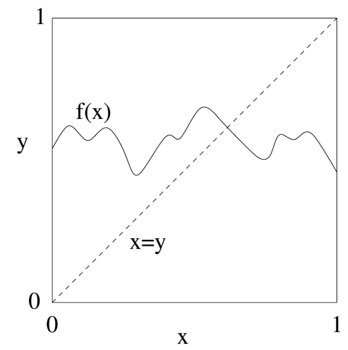
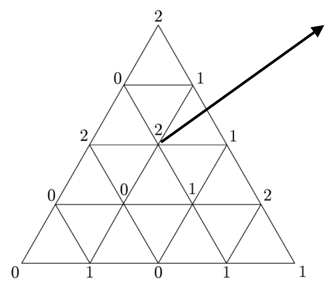
Consider the following very simple convex and compact space:



More Technical Definitions

Simplicial Subdivision or Triangulation: A triangulation of an n-simplex T is a finite set of simplicies $\{T_i\}$ for which $\bigcup_{T_i \in T} T_i = T$ and for any $T_i, T_j \in T$, either $T_i \cap T_j = \varnothing$ or $T_i \cap T_j$ is equal to a common face.



Labelling function:
$\mathscr{L} : T \to \{0,1,\ldots,n\}$

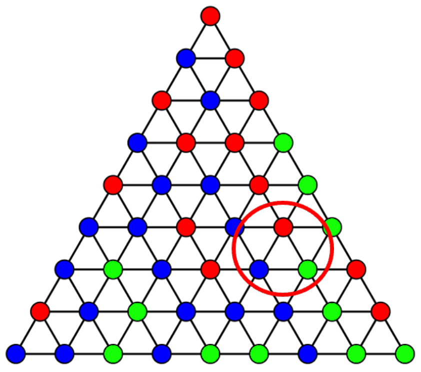
Source: Shoham and Leyton-Brown

Source: Wikipedia

Sperner's Lemma

Sperner labelling (colouring): Vertices of facet j do not receive the colour j.

Sperner's Lemma (1928): There always exists a panchromatic simplex.



Proving Brouwer via Sperner

Brouwer's Fixed Point Theorem (1911): Let $C \subset \mathbb{R}^m$ be convex and compact, and let $f : C \to C$ be a continuous function. Then f has a fixed point, i.e., there exists some point $x \in C$ such that $f(x) = x$.

Sperner's Lemma (1928): Consider a triangulation of the n-simplex coloured with a Sperner colouring. Then, there always exists a panchromatic simplex.

We will sketch the proof of Brouwer's fixed point theorem when C is the n-simplex Δ_n.

Proving Brouwer via Sperner

Let $f : \Delta_n \rightarrow \Delta_n$ be our Brouwer function.

Let $f_i(x)$ be the i'th component of f, and let x_i be the i'th component of x.

Consider a triangulation of Δ_n where the size (= distance between any two points in the same small simplex) is at most ε.

Define a labelling function $\mathscr{L}$ such that $\mathscr{L} \in \{i : f_i(x) \leq x_i\}$

It can be verified that this assigns a valid label to each point.

Intuition: If $f_i(x) > x_i$ for all i, it would hold that $\sum_i f_i(x) > \sum_i x_i = 1$, which is not possible since $\sum_i f_i(x) = 1$ also.

It can be verified that this is a valid Sperner colouring.

Proving Brouwer via Sperner

By Sperner's Lemma, we have a panchromatic simplex.

By our labelling function, that corresponds to a simplex defined by the points $(x_0, x_1, \ldots, x_n)$, such that $f_i(x_i) \leq x_i$ for each one of them.

We also know that all of these points are within distance at most ε from each other.

Take $\varepsilon \to 0$:

Intuitively, the simplex converges to a single point z, such that $f_i(z) \leq z_i$.

Actual argument uses compactness and a subsequence of centroids of the corresponding simplices (for each triangulation given by ε), and the continuity of f.

Similarly to before, this implies that $f(z) = z$ (fixed point) as otherwise we would have $1 = \sum_i f_i(z) < \sum_i z_i = 1$, a contradiction.

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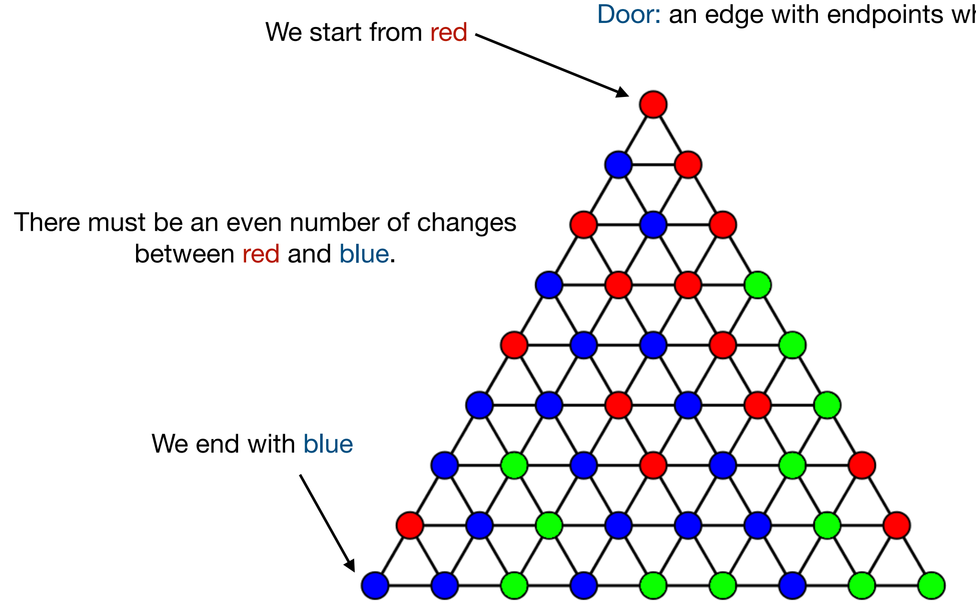
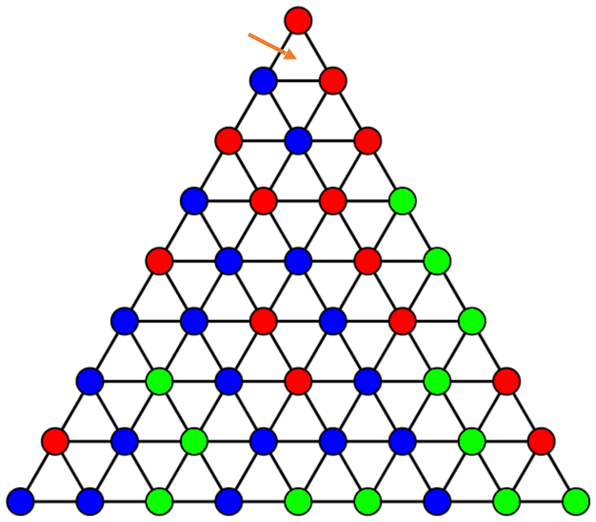
A pictorial proof of Sperner's Lemma

Door: an edge with endpoints which are **blue** and **red**.



Claim: On the boundary of Δ_n, there is an odd number of doors.

Door: an edge with endpoints which are **blue** and **red**.



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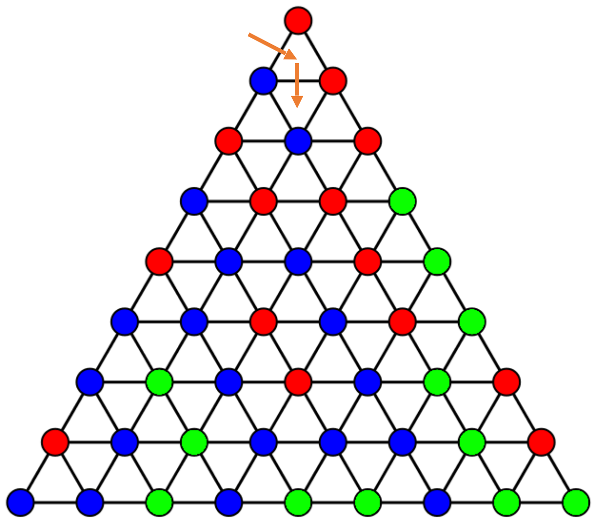

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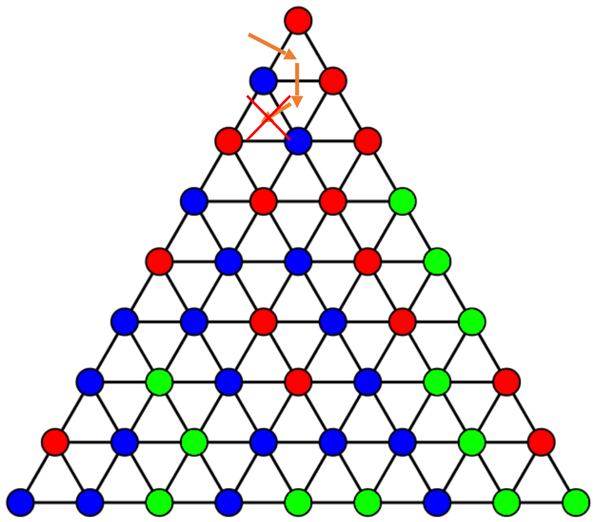

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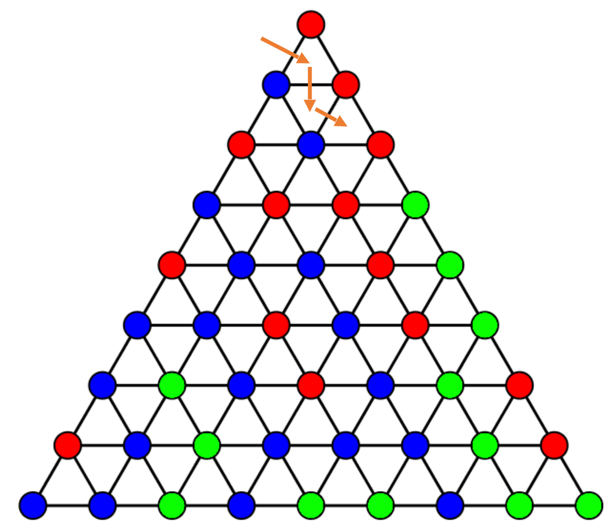

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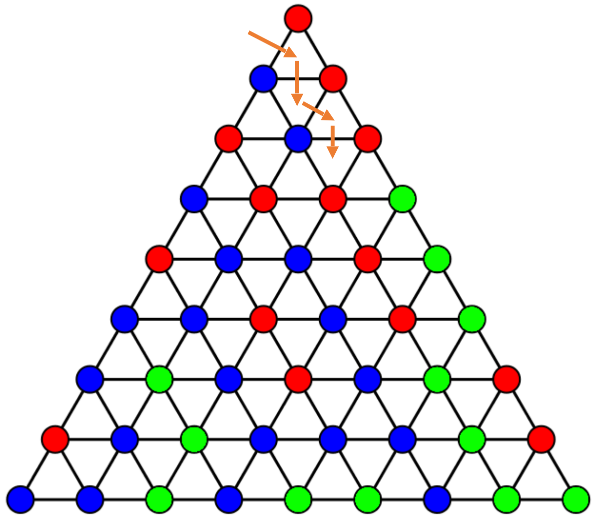

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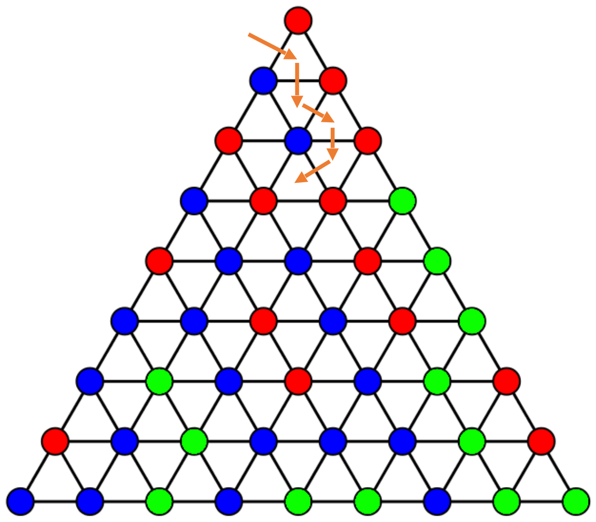

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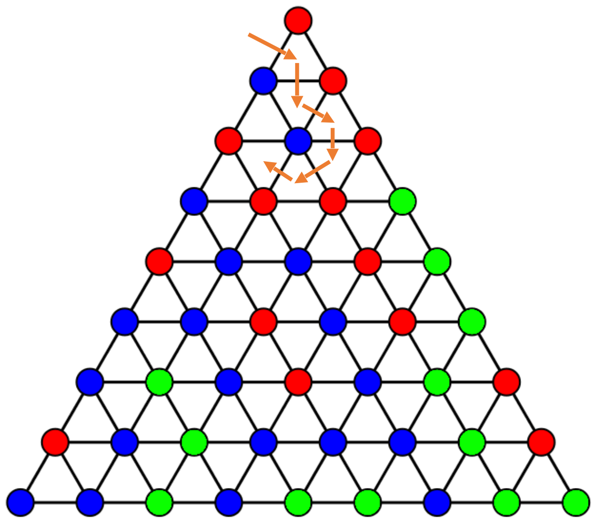

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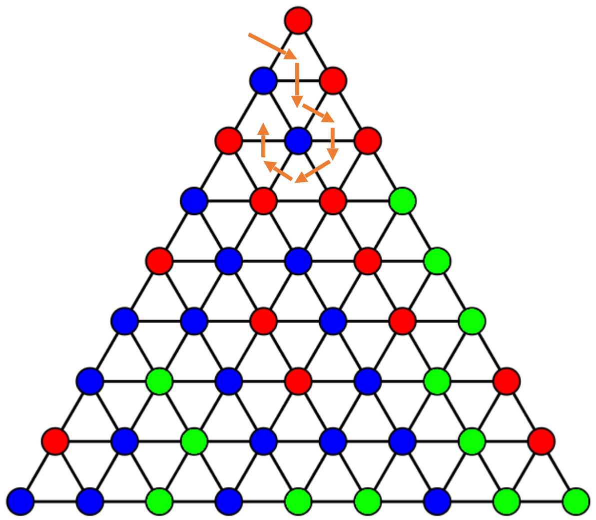
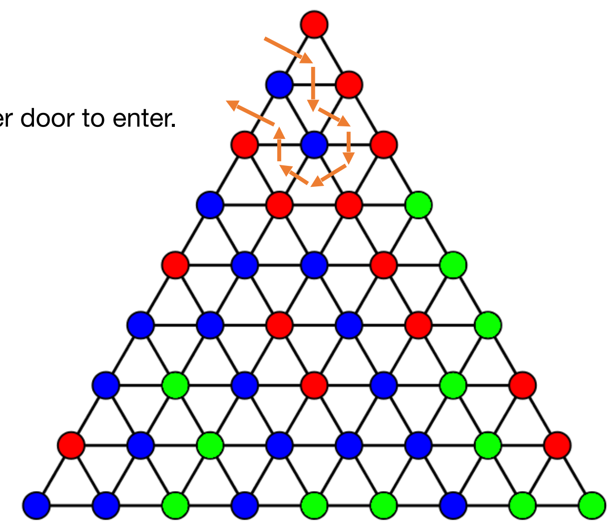


Door: an edge with endpoints which are **blue** and **red**.

There must be another door to enter.



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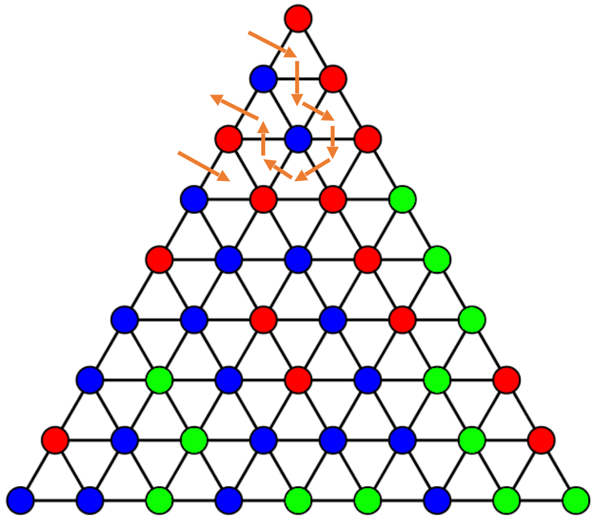

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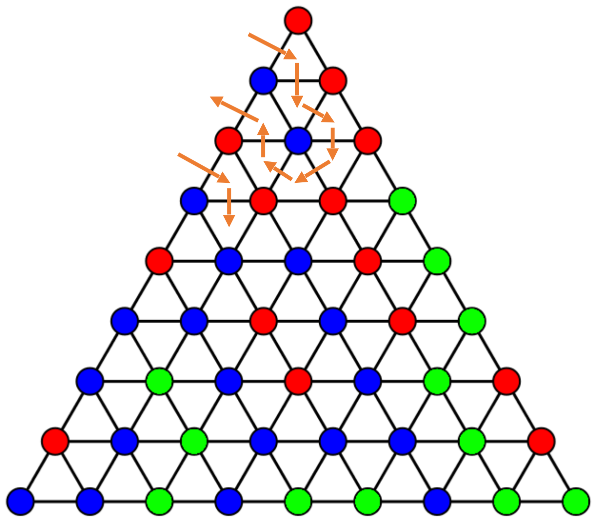

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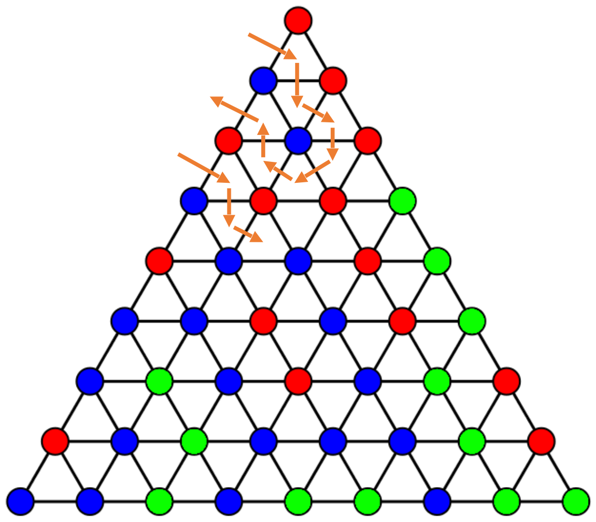

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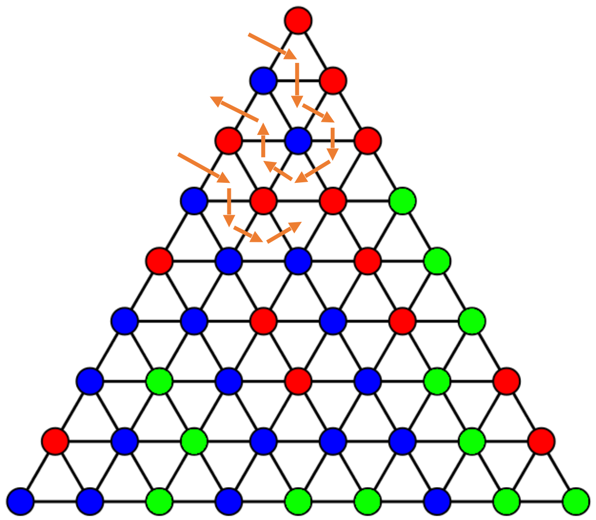

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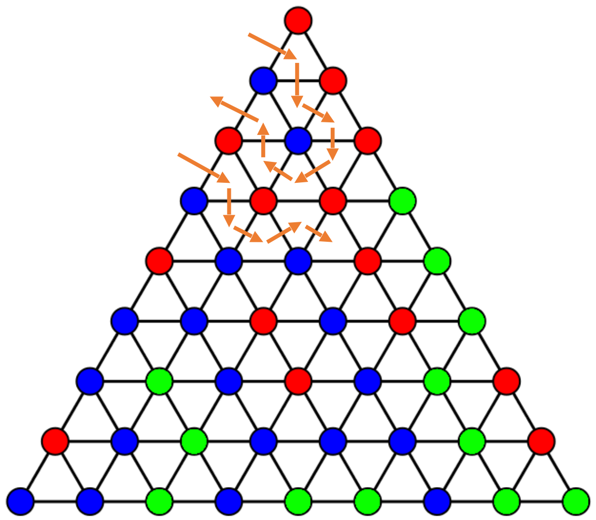

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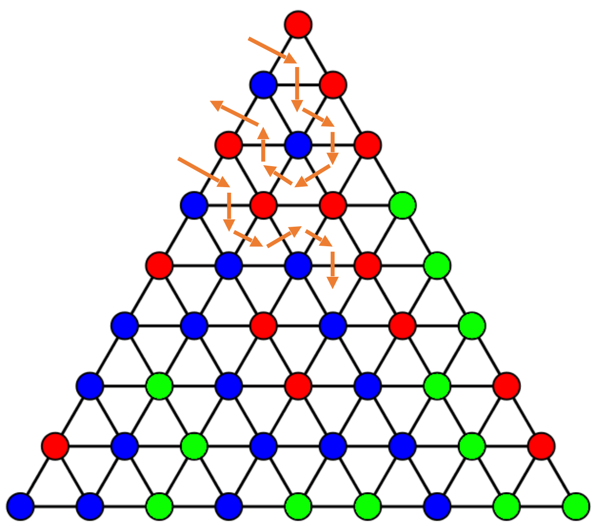

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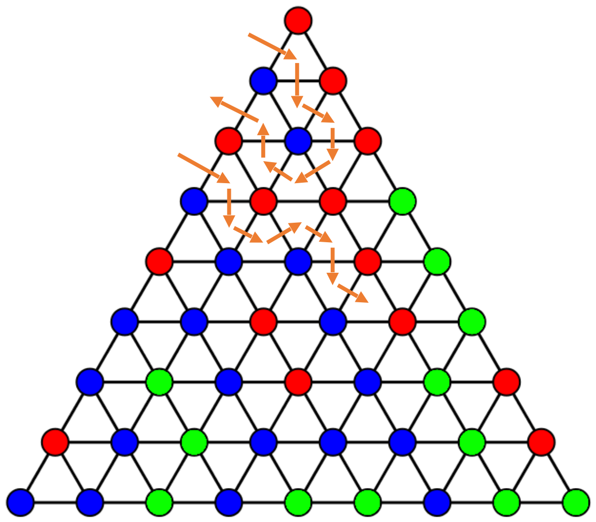

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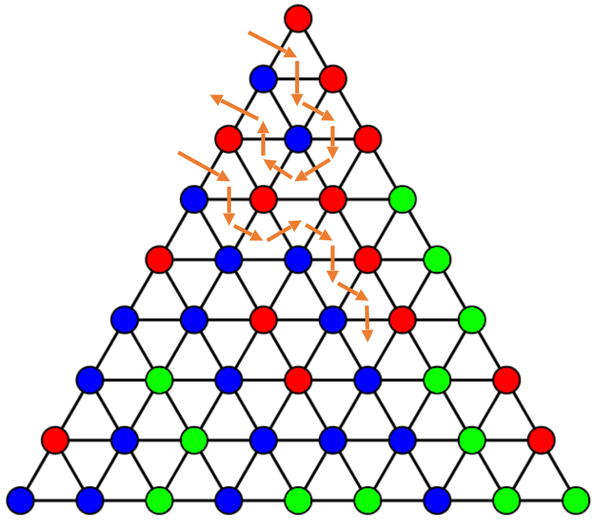

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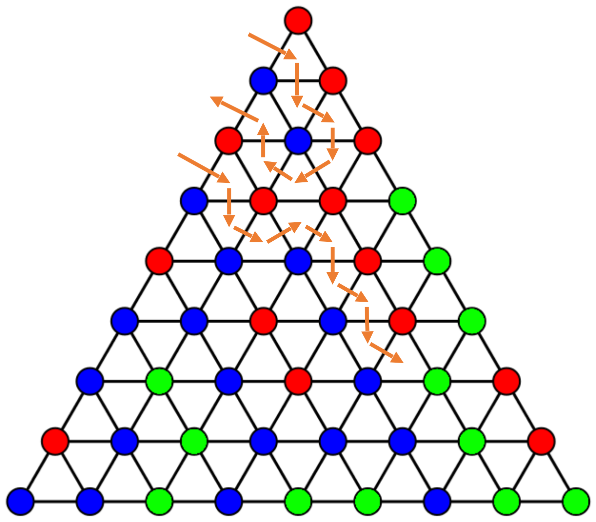
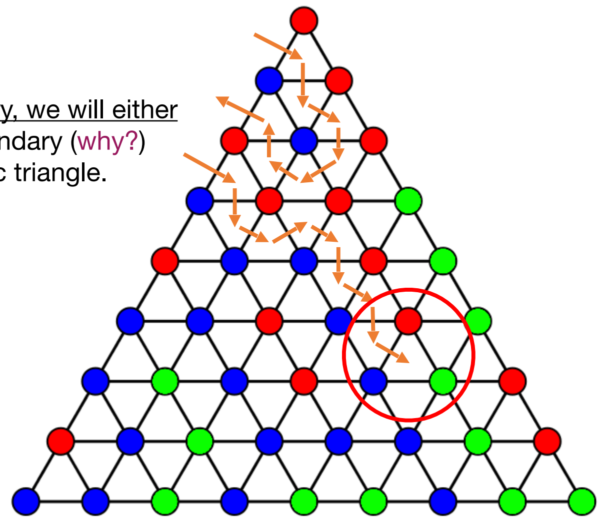


Door: an edge with endpoints which are **blue** and **red**.

<u>If we enter from the boundary, we will either</u>
(a) exist via the same boundary (why?)
(b) find a panchromatic triangle.



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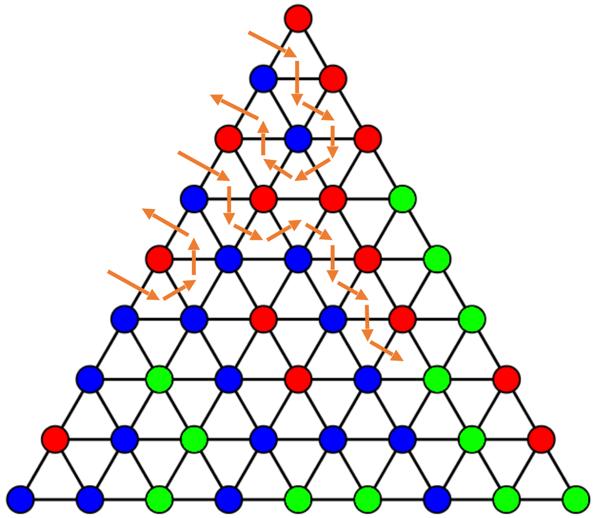

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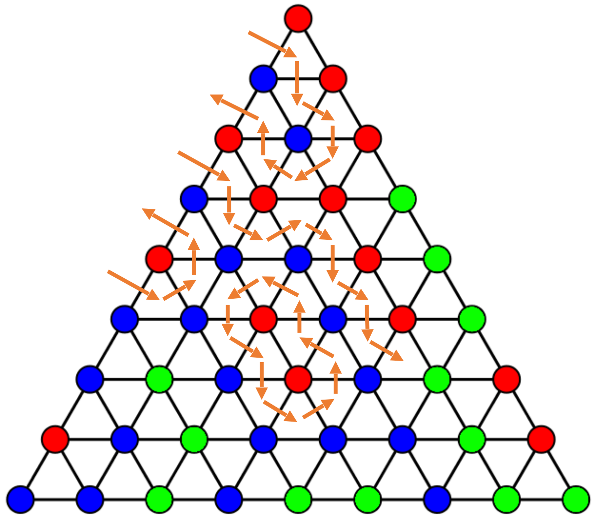

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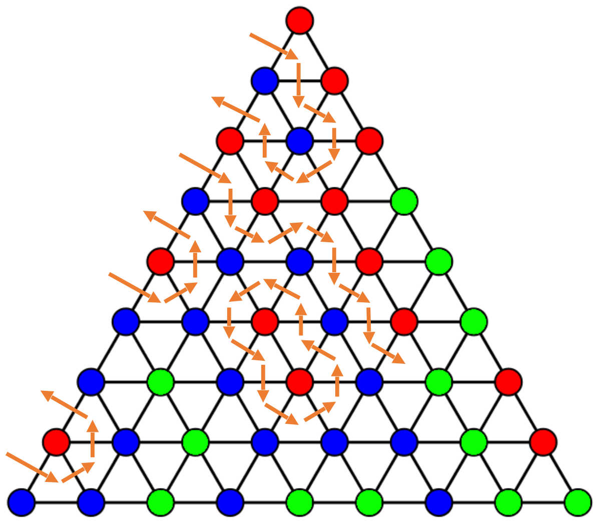

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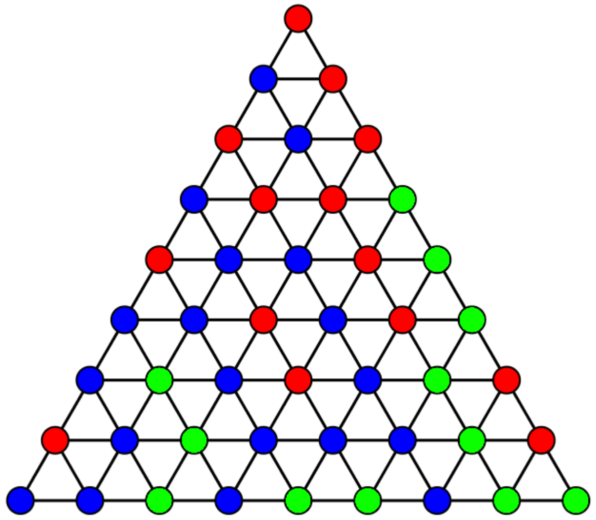
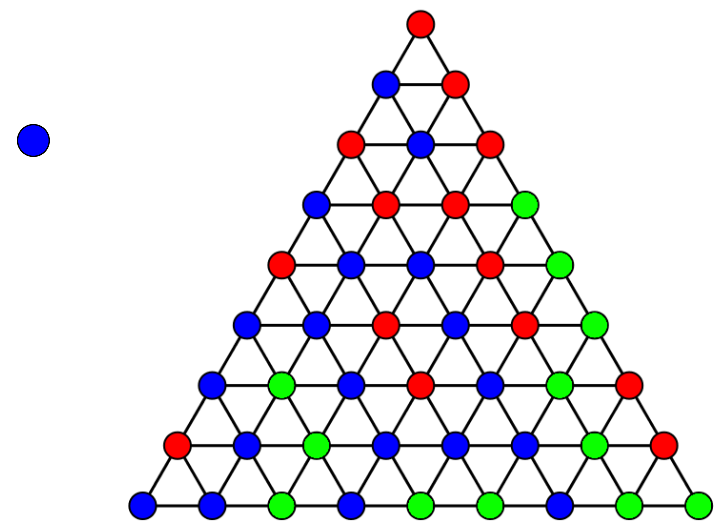
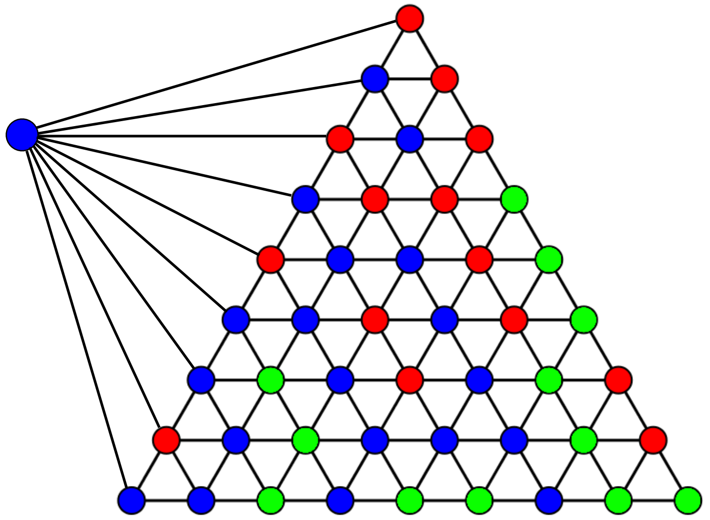
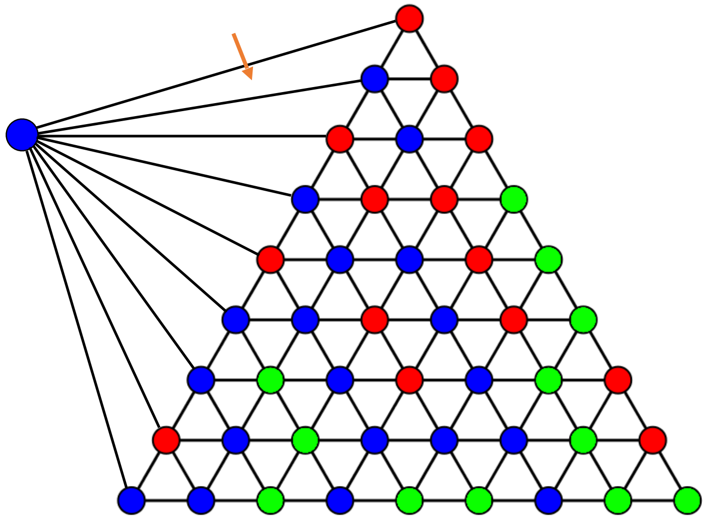


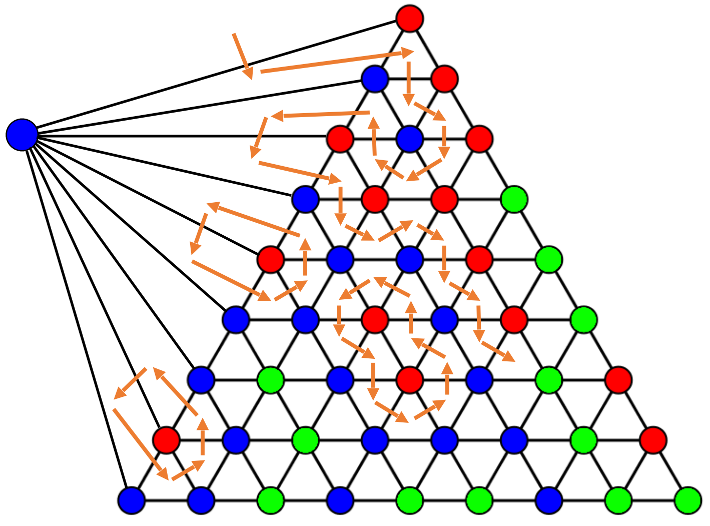
Add an artificial starting node.



Connect it with all nodes on the door boundary.

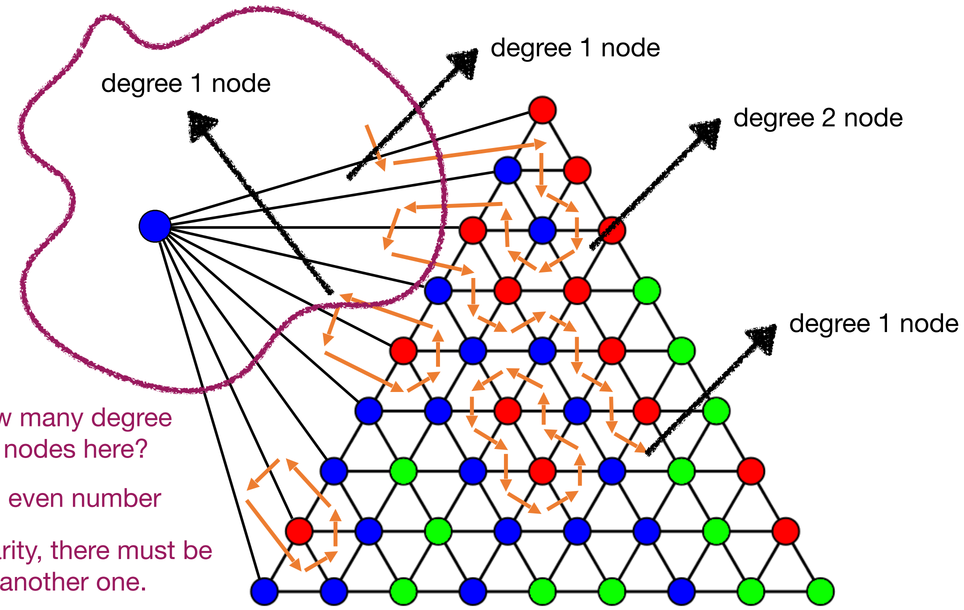






Equivalently: A graph G where the nodes are the simplices.

Edges between simplices connected via doors.



How many degree 1 nodes here?

An even number

By parity, there must be another one.

That can only be at a panchromatic triangle.

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Level 3: We will prove the lemma from topology (Sperner's Lemma) from first principles.

Nash Equilibrium Existence

Nash's Theorem (1950): Every finite, normal-form game has at least one mixed Nash equilibrium.

Next, we will see a proof (sketch) of that theorem.

We will consider different levels to the proof.

Level 1: We will prove the theorem using a theorem from topology (Brouwer's fixed point theorem) as a tool.

Level 2: We will prove the theorem from topology (Brouwer's fixed point theorem) using a different lemma from topology (Sperner's Lemma).

Level 3: We will prove the lemma from topology (Sperner's Lemma) from first principles.

Nash Equilibrium Existence

Nash's Theorem (1950): Every finite, normal-form game has at least one mixed Nash equilibrium.

If you are interested in the full proof, see Shoham and Leyton-Brown - Multiagent Systems, Chapter 3.3.4.

A more informal exposition: Roughgarden - Twenty Lectures in Game Theory, Chapters 20.4 and 20.5.1.

<u>What we didn't do:</u>

A rigorous proof of Brouwer's Theorem from Sperner's Lemma, and in fact for the case of the simplotope domain.

A rigorous proof of Sperner's Lemma in n dimensions, which uses induction.

ε-Best Responses

A pure strategy s_i of player i is a best response to the pure strategies of the other players s_{-i} if it maximises the player's utility among all possible pure strategies.

$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s'_i \in S_i$

Defined similarly for mixed strategies:

$u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i})$ for all $x'_i \in \Delta(S_i)$

ε-best response: $u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i}) - \varepsilon$

Intuition: A player can increase their utility, but not more than ε.

ε-Nash Equilibrium

A pure strategy profile s is a pure ε-Nash equilibrium, if for every player with strategy s_i in s, s_i is an ε-best response.

A mixed strategy profile x is a mixed ε-Nash equilibrium, if for every player with strategy x_i in x, x_i is an ε-best response.

Why ε-Nash Equilibria?

Conceptual Motivation: If a player cannot increase their utility by much, they will not bother deviating $\Rightarrow$ ε-Nash equilibria are still quite robust, especially when ε is very small.

Computational Motivation: Nash equilibria (i.e,, with $\varepsilon = 0$) might require *irrational numbers* to be described.

e.g., maybe some strategy needs to be played with probability $1/\sqrt{5}$.

How are we going to represent those equilibria on our computer, which can only use rational numbers?

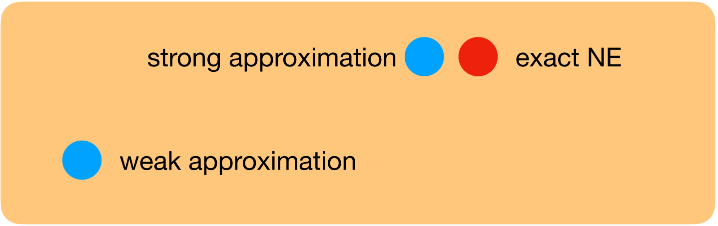
An important remark

A mixed strategy profile x is a mixed ε-Nash equilibrium, if for every player with strategy x_i in x, x_i is an ε-best response.

An important remark

A mixed strategy profile σ is a mixed ε-Nash equilibrium (weak approximation), if for every player with strategy x_i in x, x_i is an ε-best response.

A mixed strategy profile x is a mixed ε-Nash equilibrium (strong approximation), if x^* is some (exact) mixed Nash equilibrium and $||x - x^*||_\infty \leq \varepsilon$.



Another important remark

Conceptual Motivation: If a player cannot increase their utility by much, they will not bother deviating $\Rightarrow$ ε-Nash equilibria are still quite robust, especially when ε is very small.

Computational Motivation: Nash equilibria (i.e,, with $\varepsilon = 0$) might require *irrational numbers* to be described.

e.g., maybe some strategy needs to be played with probability $1/\sqrt{5}$.

it has been shown that this might be the case when there are 3 or more players.

for 2 players, there always exist mixed Nash equilibria in *rational numbers*, as we saw, via the Lemke-Howson algorithm.

Finding Nash Equilibria for 2 players

2-NASH: Given as input a normal-form game with 2 players, with all the parameters (strategy sets, utilities) given in binary representation, return a Nash equilibrium, with the corresponding probabilities represented in binary.

Is there a class of games for which we can solve 2-NASH in polynomial time? How?

Finding Nash Equilibria

So, for three or more players, we have the following problem:

n-NASH(ε): Given as input a normal-form game with n players, with all the parameters (strategy sets, utilities) given in binary representation, and an $\varepsilon > 0$, return an ε-Nash equilibrium, with the corresponding probabilities represented in binary.

Polynomial time algorithms?

Can we design polynomial time algorithms for either
2-NASH or n-NASH(ε)?

Complexity of MNE computation

If the answer is yes, the evidence is such an algorithm.

If the answer is no, what is the evidence?

Computational hardness.

NP-hardness: Informally, we should not expect to find polynomial time algorithms for NP-hard problems.

Complexity of MNE computation

Is NP the right class for MNE computation? (call the problem NASH)

Some NP-hard problems:

- SAT: Given a boolean formula in CNF form, *decide whether there exists* a satisfying assignment.

- VERTEX COVER: Given a graph G and a number k, *decide whether there exists* a vertex cover of size at most k in G.

Is NASH different?

Given a game G, decide if there exists a MNE?

This is trivial.

Given a game G, *find* a MNE, *which we know exists*.

The Class TFNP

Defined by (Megiddo and Papadimitriou 1988).
"Total Search Problems in NP"

Total: A solution is guaranteed to exist.
Search: We are looking for a solution.
- ➢e.g., find a Nash equilibrium

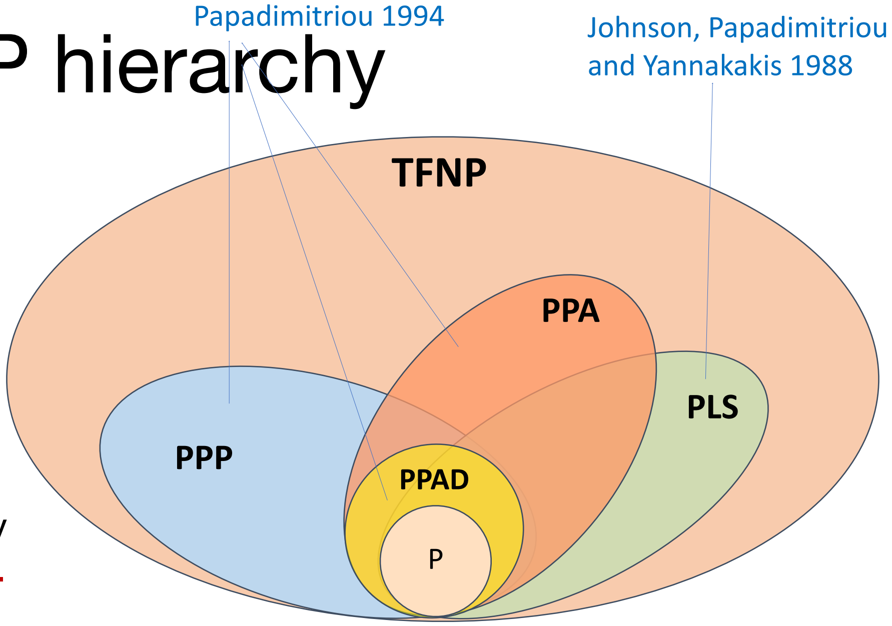
in NP: Given a candidate solution, we can verify it in polynomial time.

The TFNP hierarchy

Define several subclasses of TFNP.

Show completeness results for those classes.

Approach initiated by (Papadimitriou 1994).



PPAD
(Polynomial Parity Argument on a Directed Graph)

END-OF-LINE:
Input: An exponentially large directed graph, implicitly given as input, with vertices of indegree and outdegree at most 1.
A vertex of indegree zero (a source).

Output: Another vertex of indegree 0 or a vertex of outdegree 0 (another source or a sink)

Two polynomial-sized circuits P and S that input a vertex and output its predecessor and its successor respectively.

PPAD
(Polynomial Parity Argument on a Directed Graph)

PPAD membership: A problem is in PPAD if it can be reduced to END-OF-LINE in polynomial time.

PPAD-hardness: A problem is PPAD-hard if END-OF-LINE can be reduced to it in polynomial time.

PPAD-completeness: PPAD membership + PPAD-hardness.

Complexity of MNE computation

Theorem (Chen and Deng 06): 2-NASH is PPAD-complete.

Theorem (Goldberg, Daskalakis, and Papadimitriou 06): n-NASH is PPAD-complete.

These results essentially mean that we should not hope to design polynomial time algorithms for finding MNE in games *in general*, and this is inherently a hard computational problem.