Algorithmic Game Theory and Applications

Inefficiency of Equilibria

A First Game: Prisoner's Dilemma

Both players confessing is the *logical* outcome of this game.

Player 1 \ Player 2	Confess	Silent
Confess	5, 5	9,0
Silent	0, 9	8, 8

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Confess	5	9
Silent	0	8

Player 1 (row player)

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Player 2 (column player)

For Player 1, confessing is better regardless of the strategy of Player 2

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Notions of Efficiency

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Social Welfare (in a game with utilities): The (expected) social welfare of a strategy profile $x = (x_1, \ldots, x_n)$ is the sum of utilities of all the players, i.e.,

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Social Cost (in a game with costs): The (expected) social cost of a strategy profile $x = (x_1, \ldots, x_n)$ is the sum of utilities of all the players, i.e.,

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In fact, in most cases we can assume that this strategy profile is pure, therefore s.

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We can have this actually for any solution concept, e.g., "correlated Price of Anarchy" for *correlated equilibria* (Tutorial 3).

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Price of Anarchy of a class of games

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For cost minimisation games

$$\text{PoA}\ (\mathcal{G}) = \max_{G \in \mathcal{G}} \frac{\max_{x \in \text{MNE}(G)} \text{SC}(x)}{\text{SC}(x^*)},$$

where $x^* \in \arg\min_x \text{SC}(x)$ and $\text{MNE}(G)$ is the set of mixed Nash equilibria of the game G.

We flip the ratio to maintain the convention that $\text{PoA} \geq 1$ always.

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Maybe we can be more optimistic: what if we consider the best possible equilibrium instead?

Price of Stability (Anshelevich et al. 2006).

Definitions Lookup

<u>utilities, social welfare</u>

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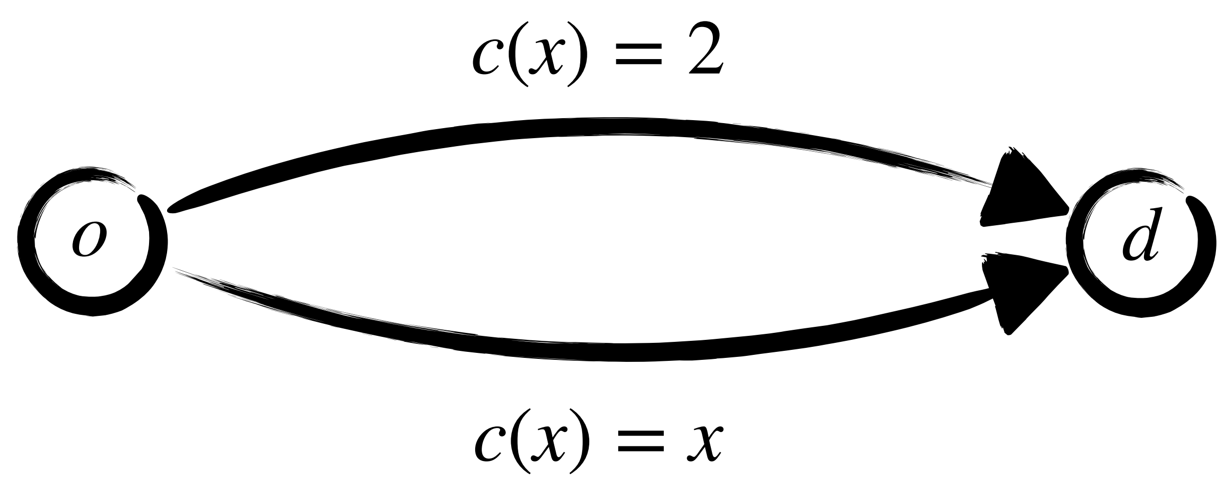
Back to congestion games



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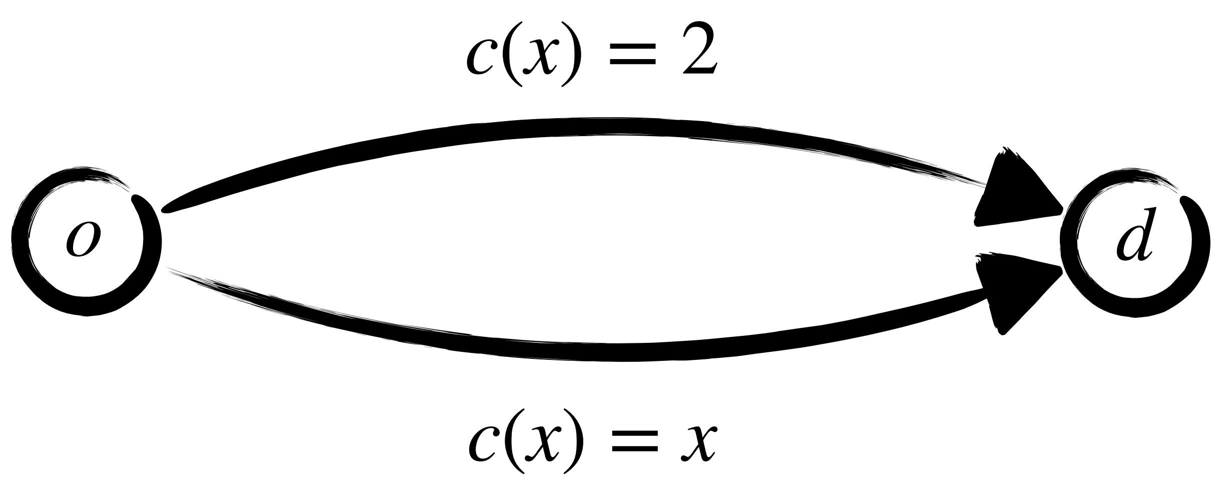
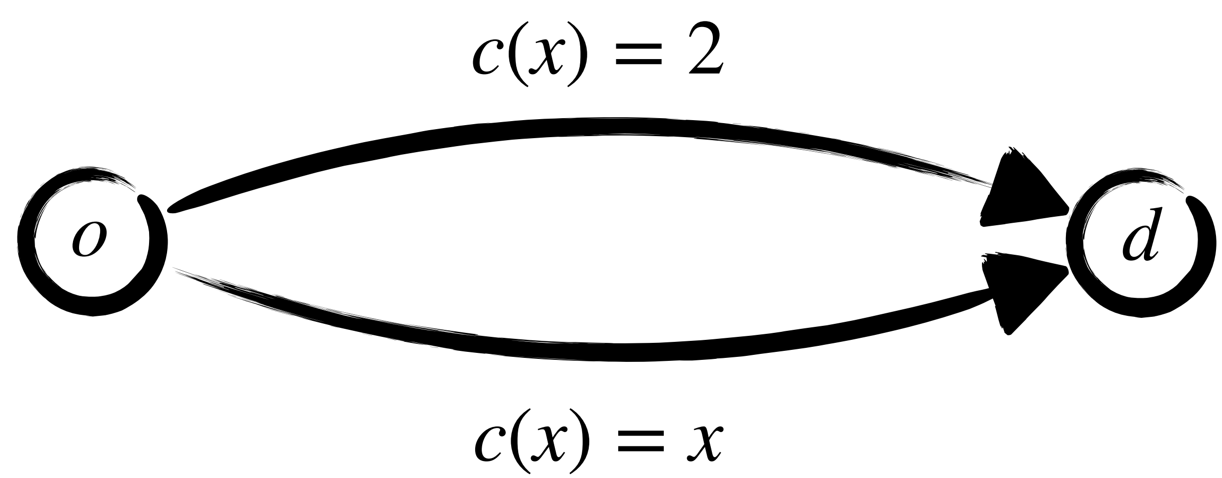


Suppose there are two players.

Back to congestion games



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What is the optimal outcome here, i.e., the one the minimises the social cost?

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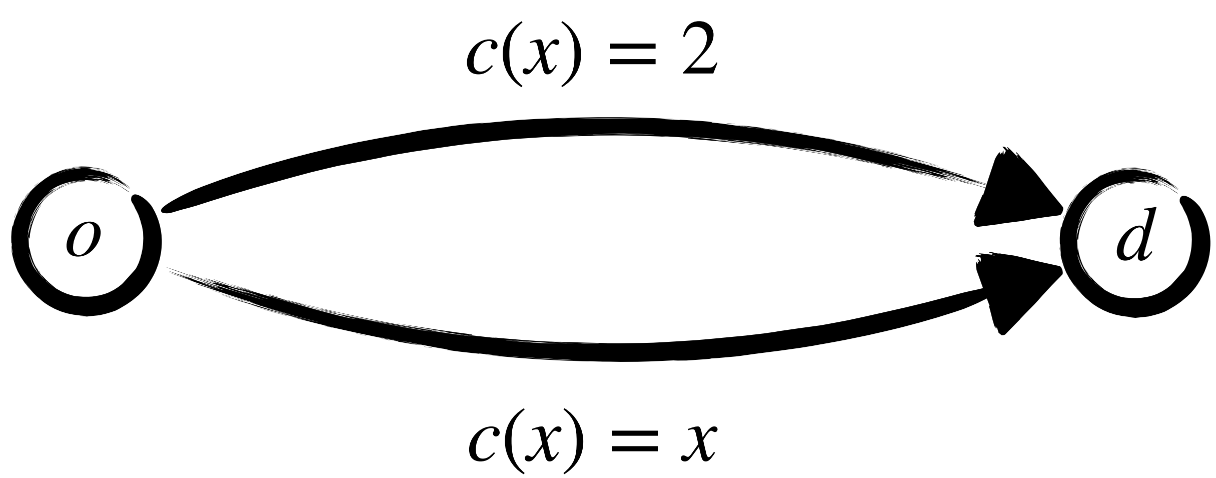

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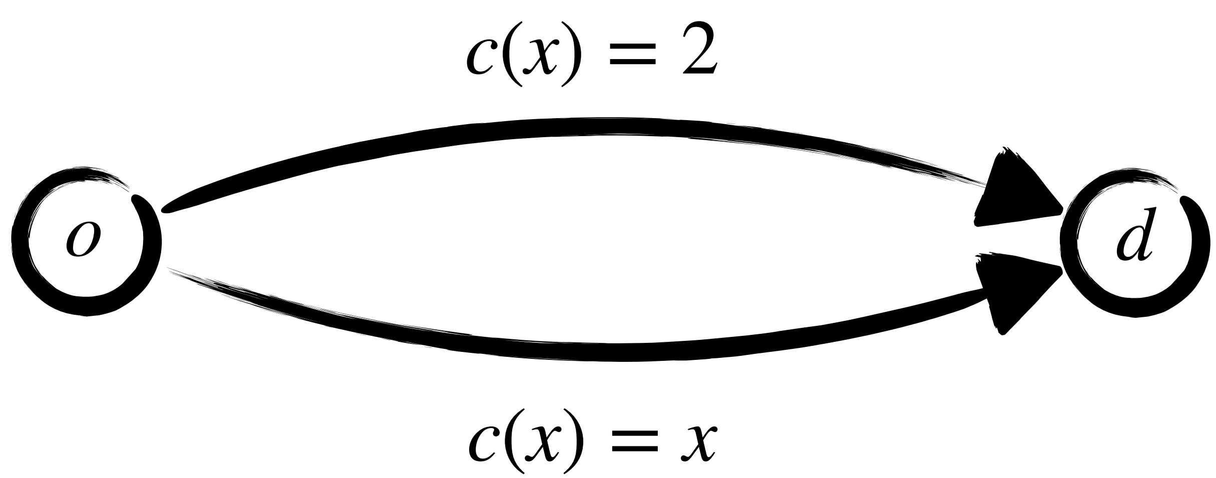

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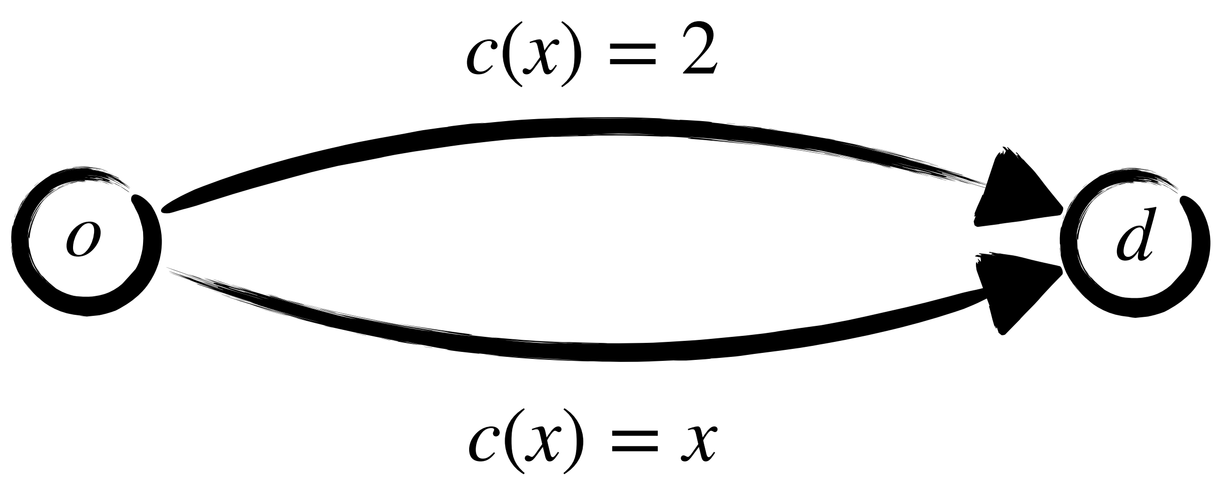

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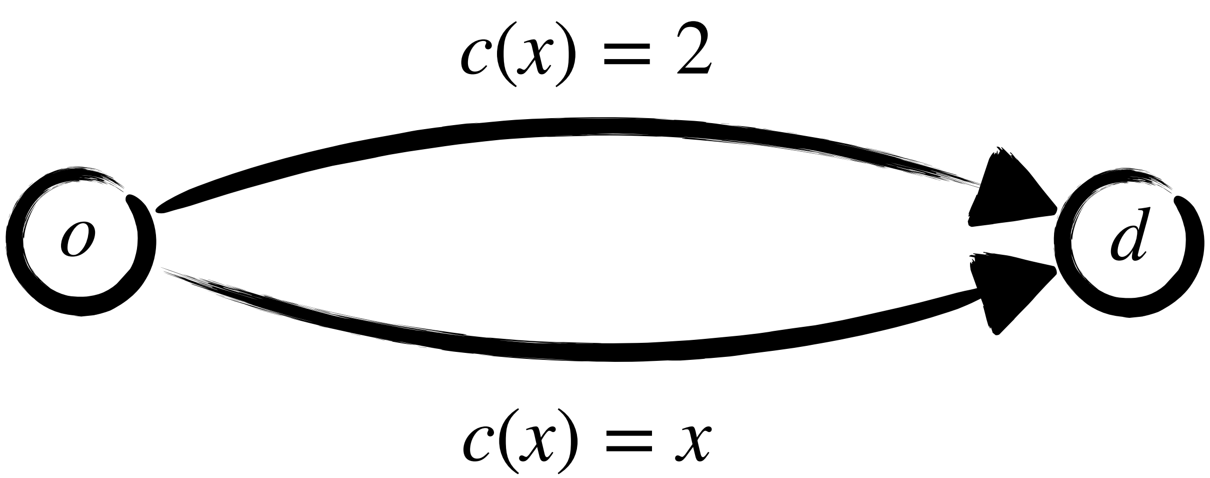

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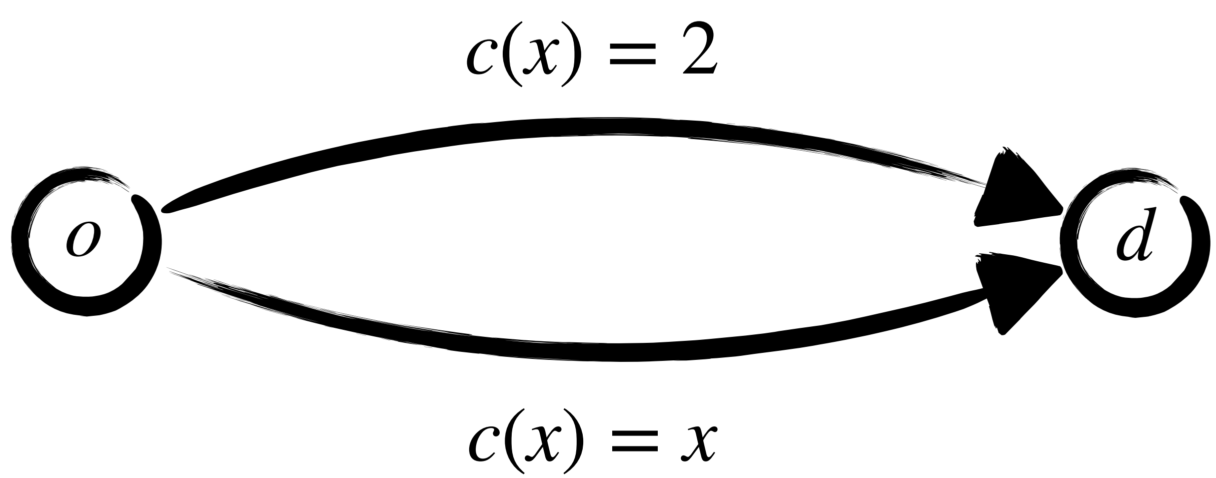

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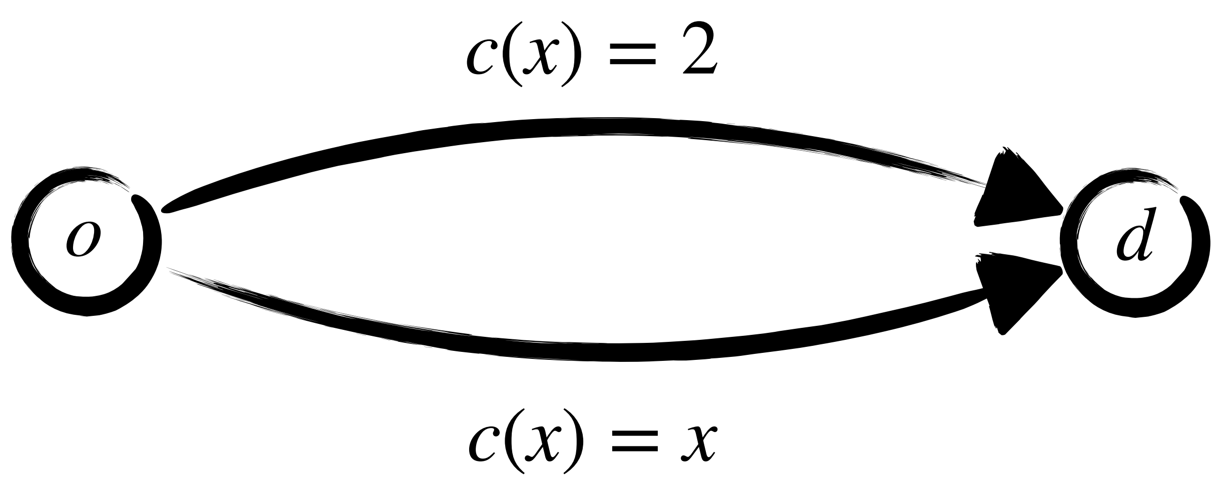

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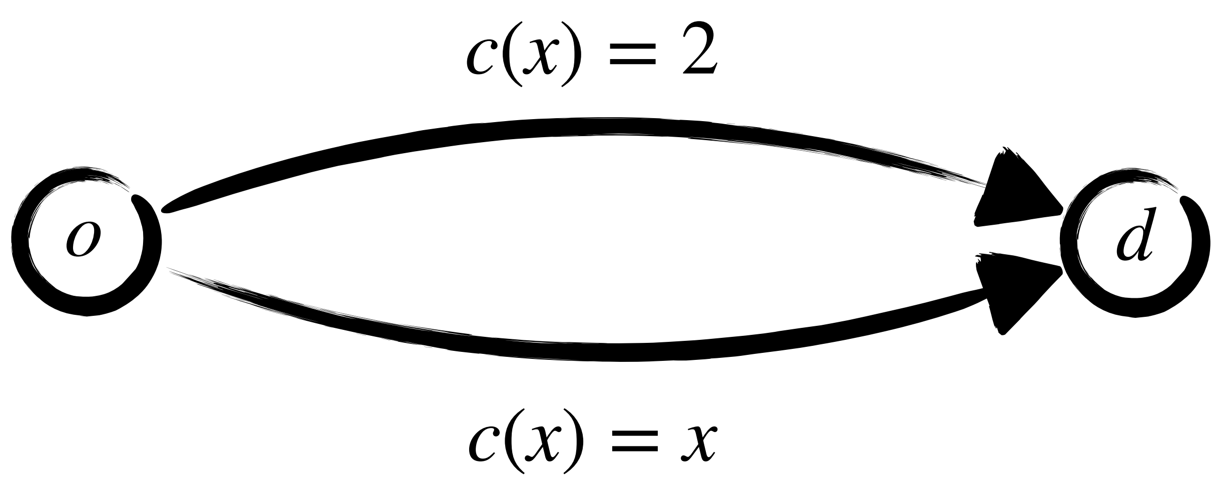

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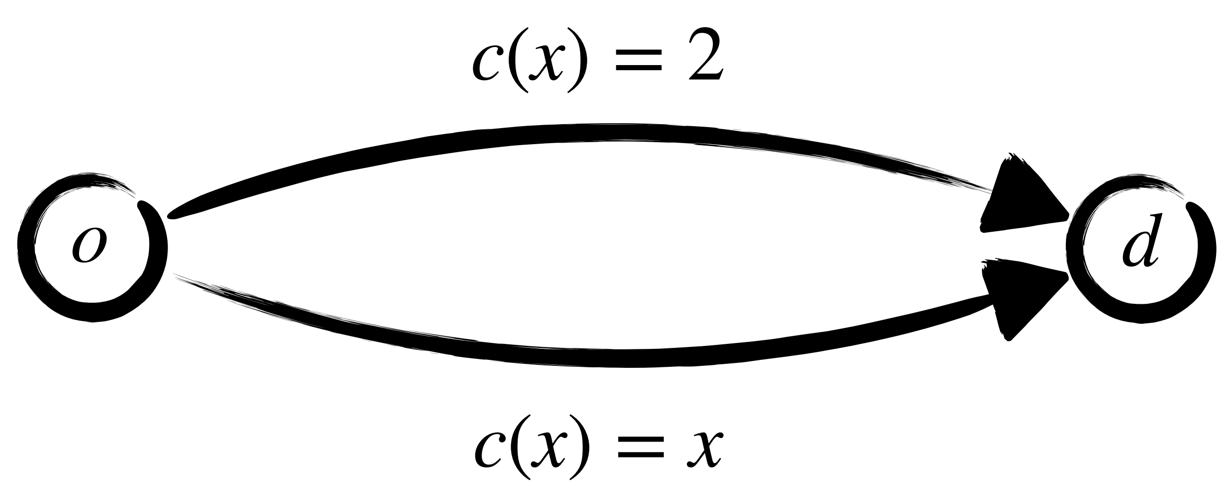

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What is the Price of Stability of the game?

Atomic Network Congestion Games

Definition: An (atomic) network congestion game is a congestion game in which the resources are edges in a directed graph, and each player must choose a set of edges that forms a (simple) path from a given source s_i to a given sink t_i.

On every edge there e is a cost function $c_e(x)$ which is a function of the number of players that have e in their chosen paths.

For example: $c_e(x)$ could be a linear function

$$c_e(x) = \alpha_e x + \beta_e$$

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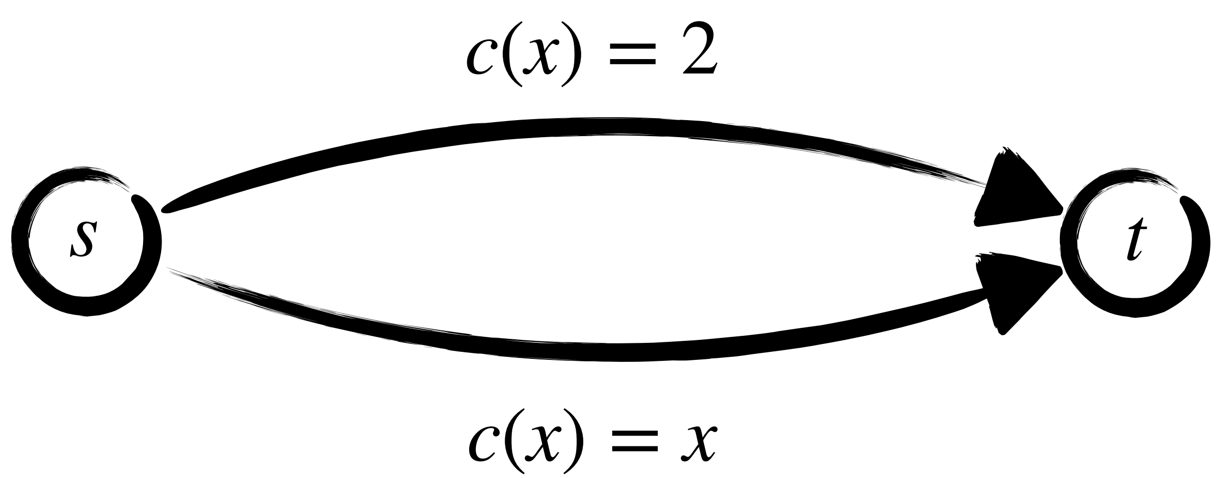

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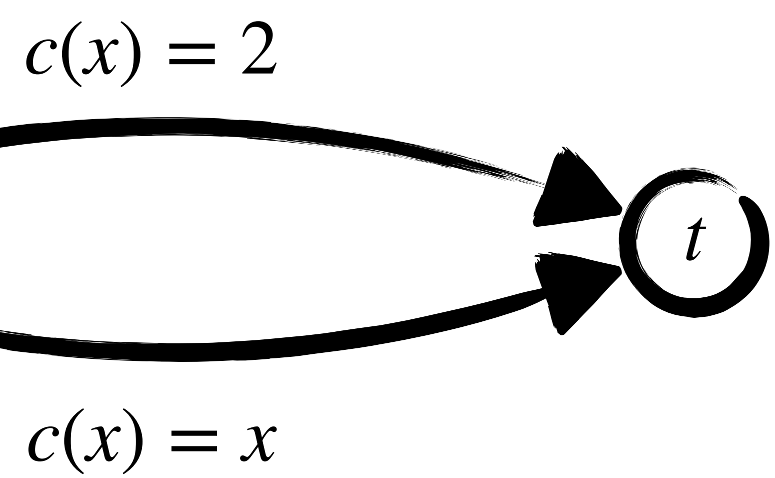

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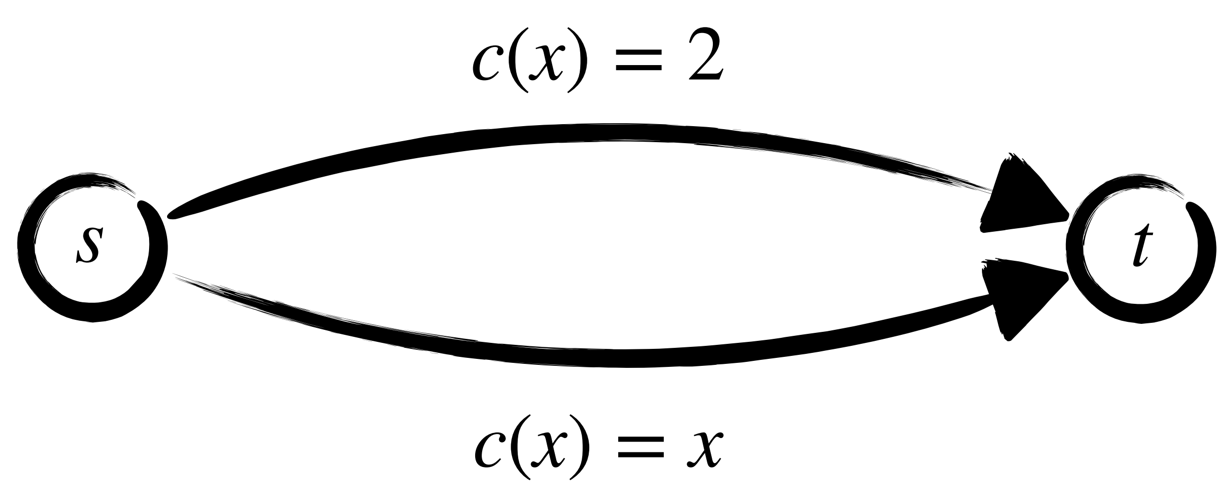

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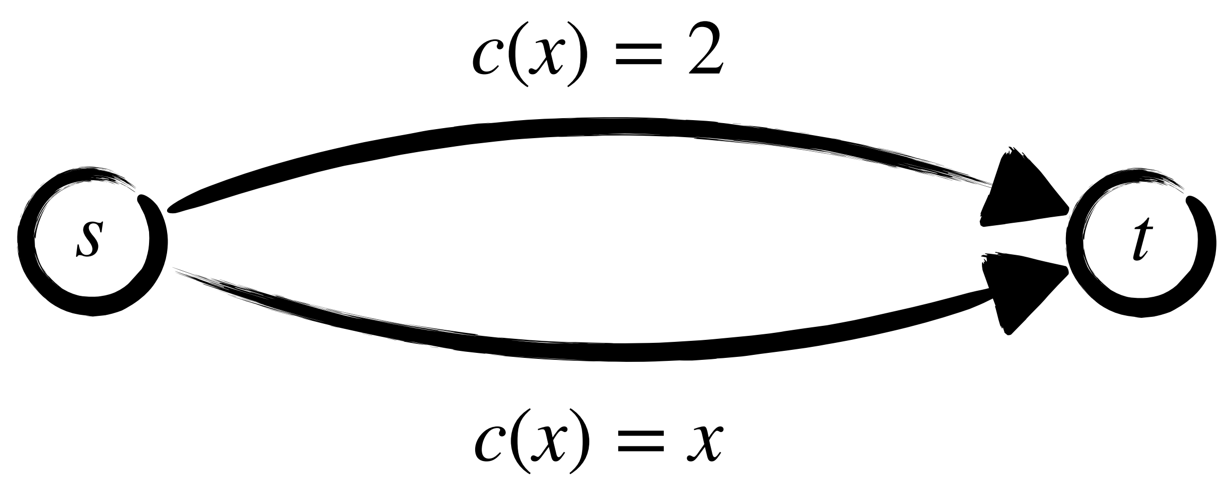

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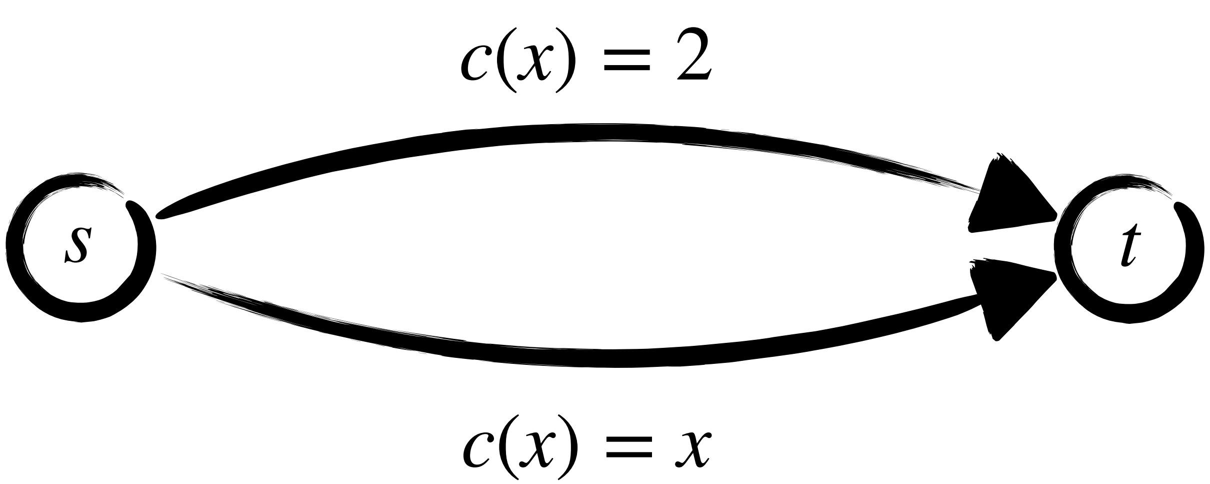
PoA $(\mathscr{G}_{\text{NC}}) \geq 4/3$

Back to congestion games



What is the optimal outcome here, i.e., the one the minimises the social cost?

s_1 = top, s_2 = bottom, SC$(s_1, s_2) = 3$

What is an equilibrium of this game?

The optimal solution is an equilibrium!

Any other equilibria?

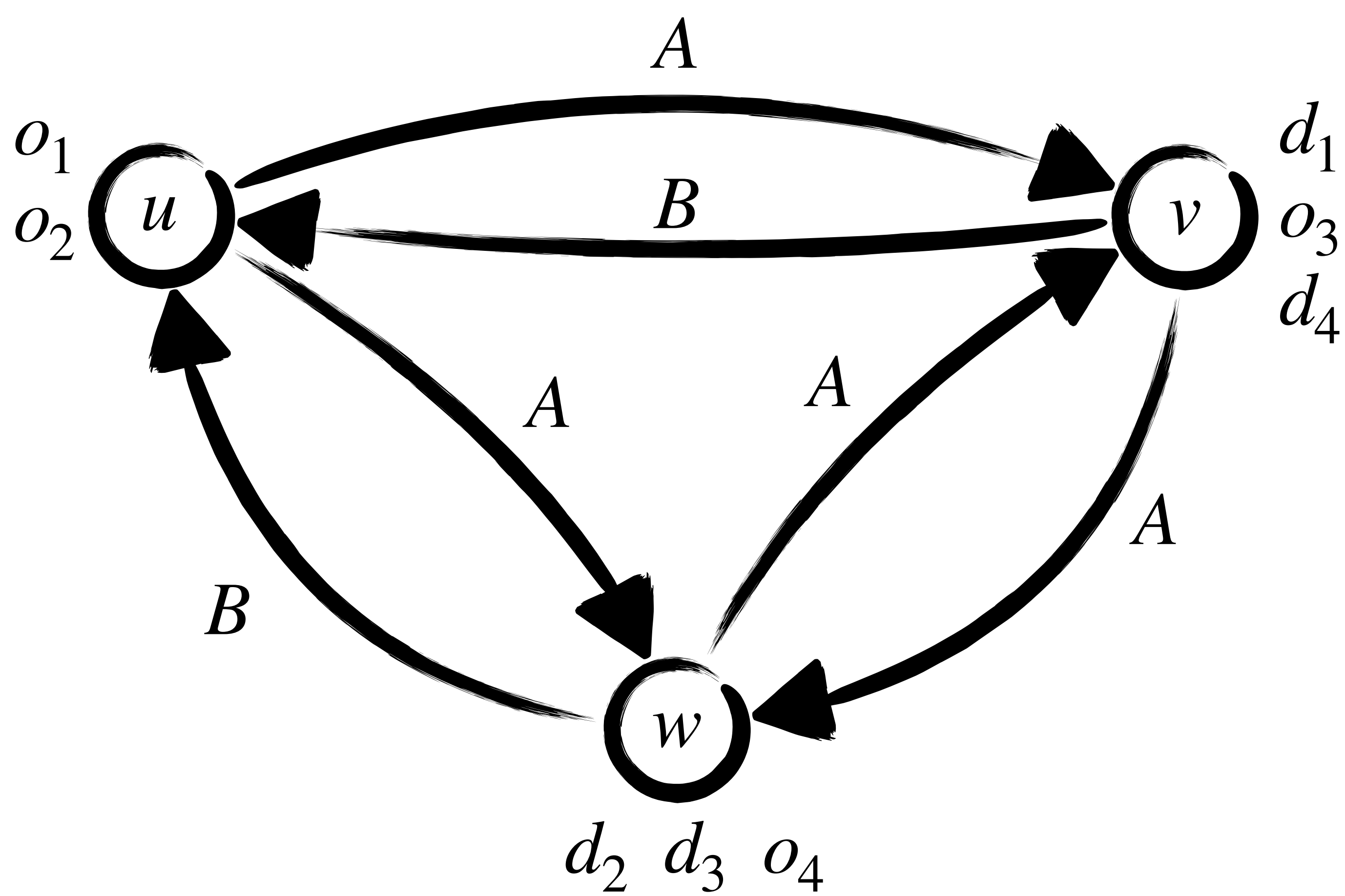
s_1 = bottom, s_2 = bottom, SC$(s_1, s_2) = 4$

What can we say about the PoA / PoS of network congestion games?

PoA $(\mathcal{G}_{\text{NC}}) \geq 4/3$

PoS $(\mathcal{G}_{\text{NC}}) \geq 1$

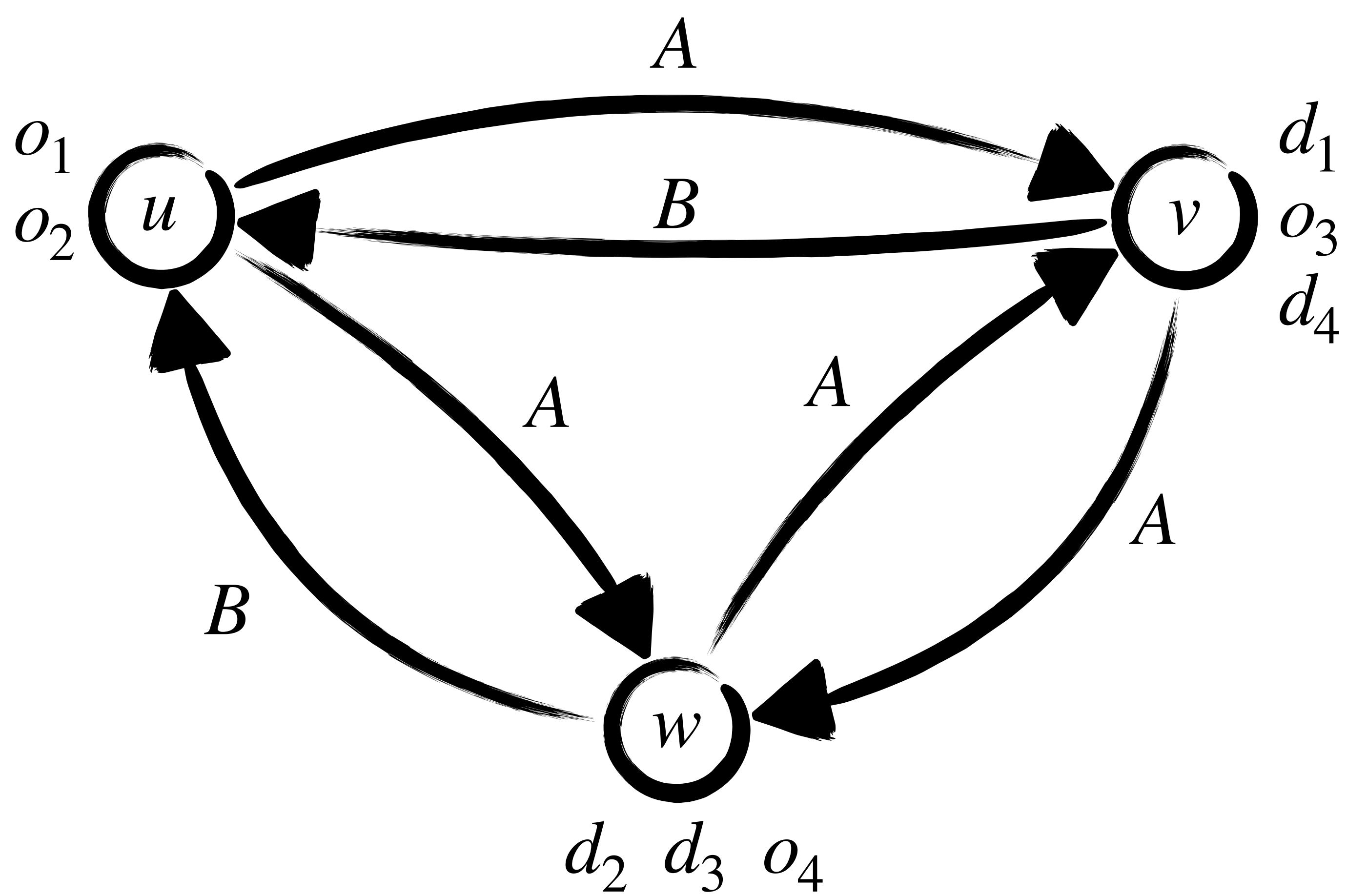
Another network congestion game



$A\,.\ c(x) = x$

$B\,.\ c(x) = 0$

Another network congestion game



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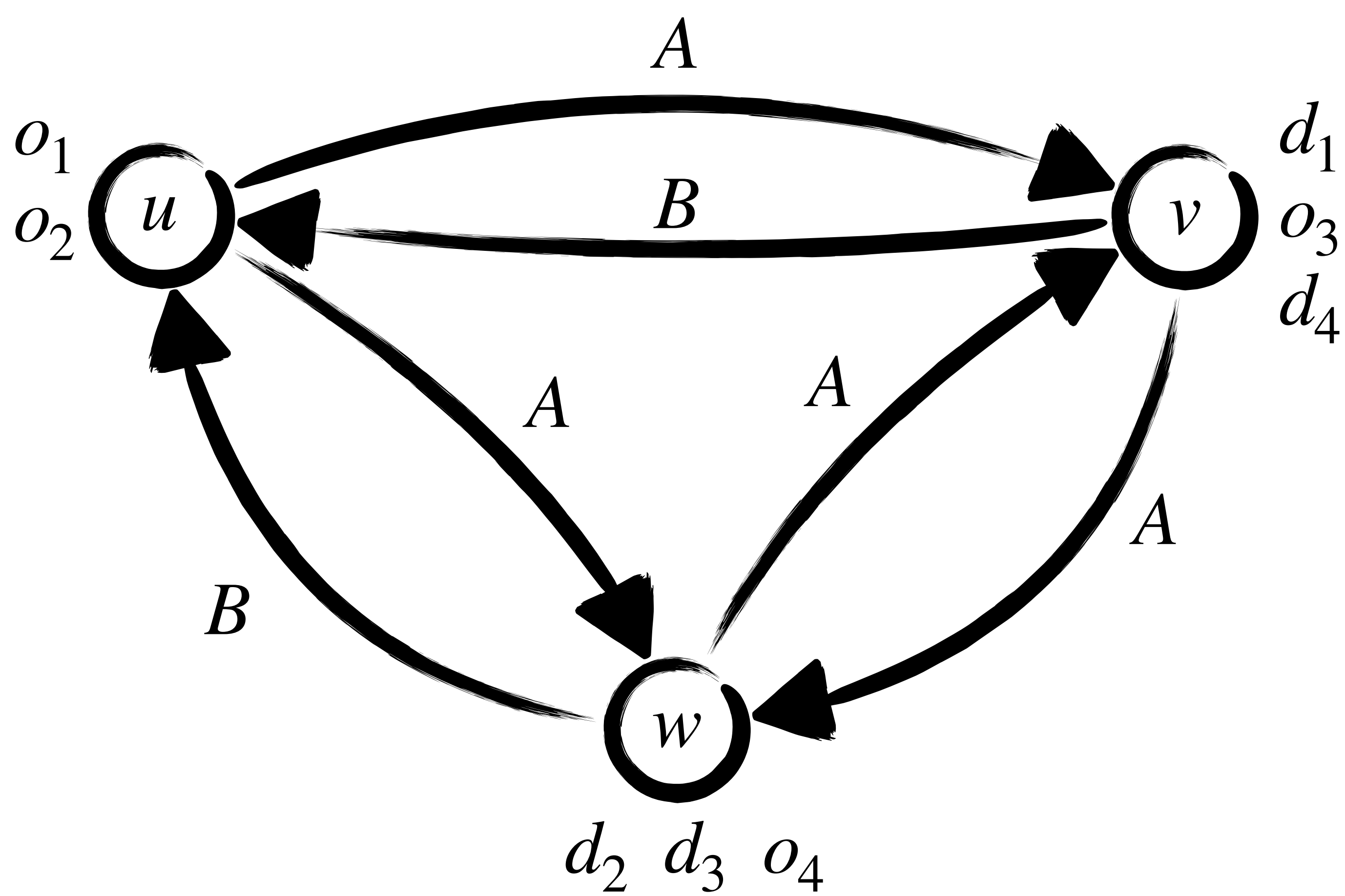
What is the optimal outcome?

Another network congestion game

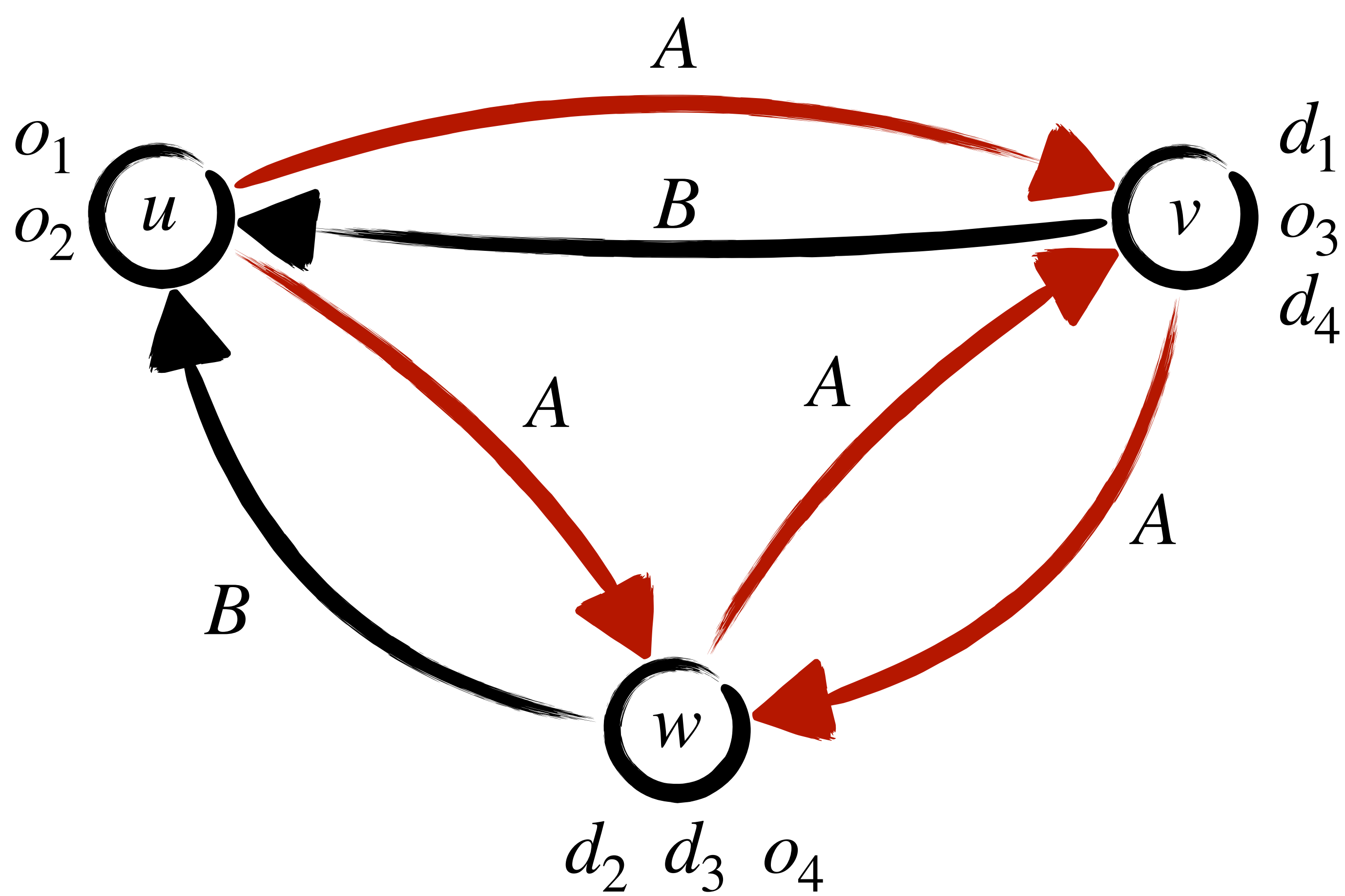


A . $c(x) = x$

B . $c(x) = 0$

What is the optimal outcome? Every player takes the one-hop path.

Another network congestion game



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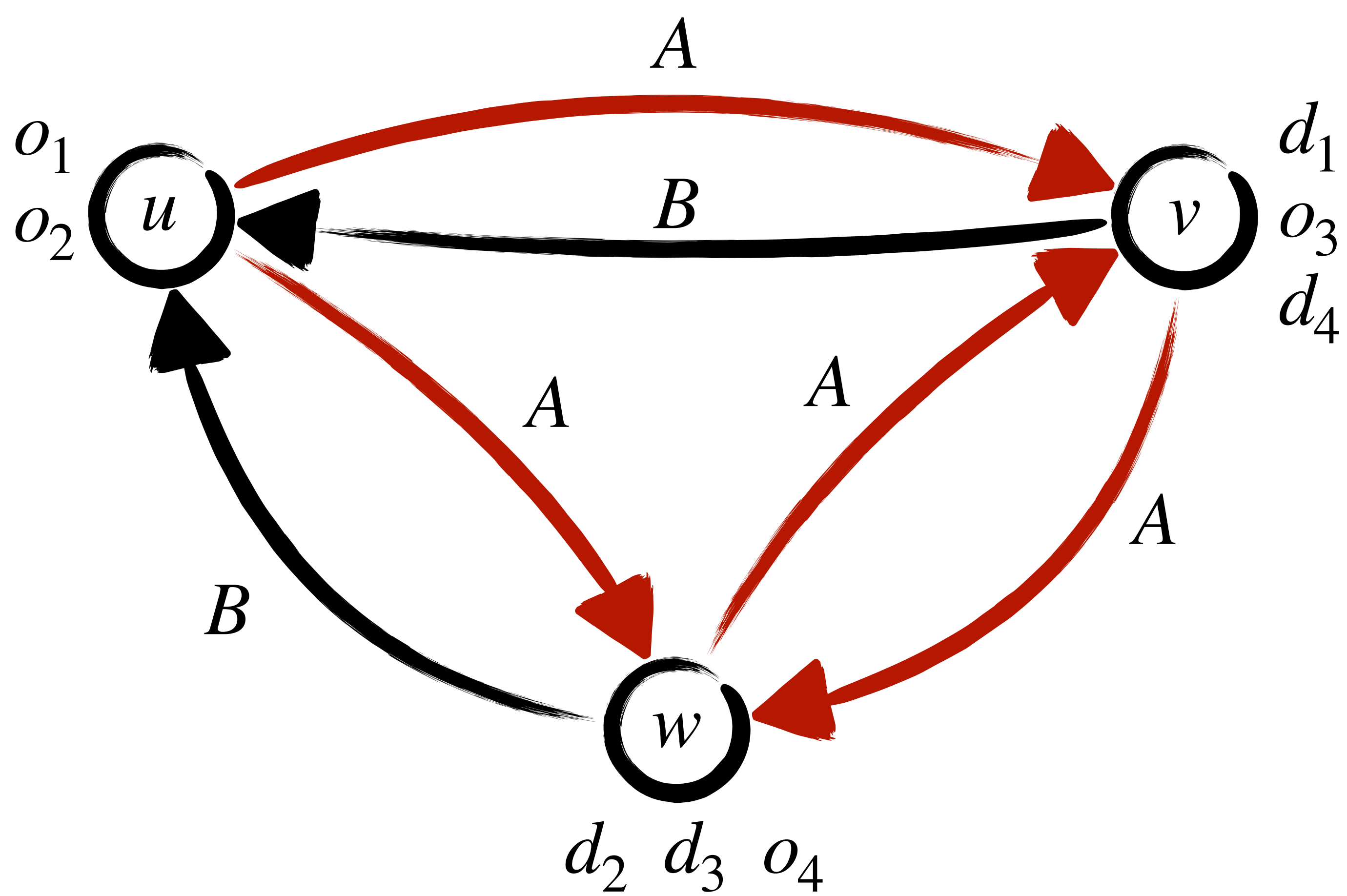


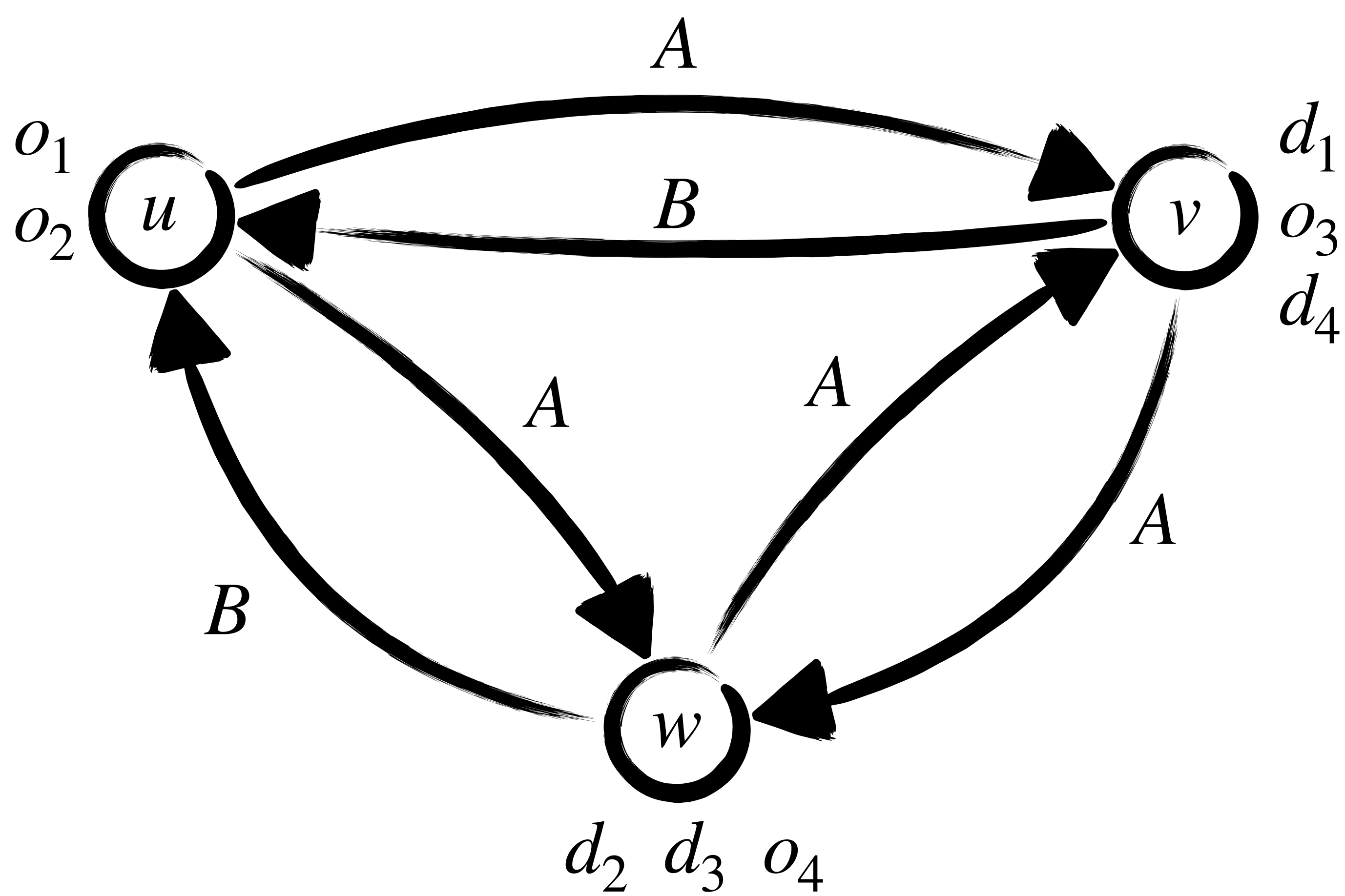
$A\,.\;c(x) = x$

$B\,.\;c(x) = 0$

What is the optimal outcome? Every player takes the one-hop path.

SC(one-hop, one-hop, one-hop) $= 1 + 1 + 1 + 1 = 4$

Another network congestion game



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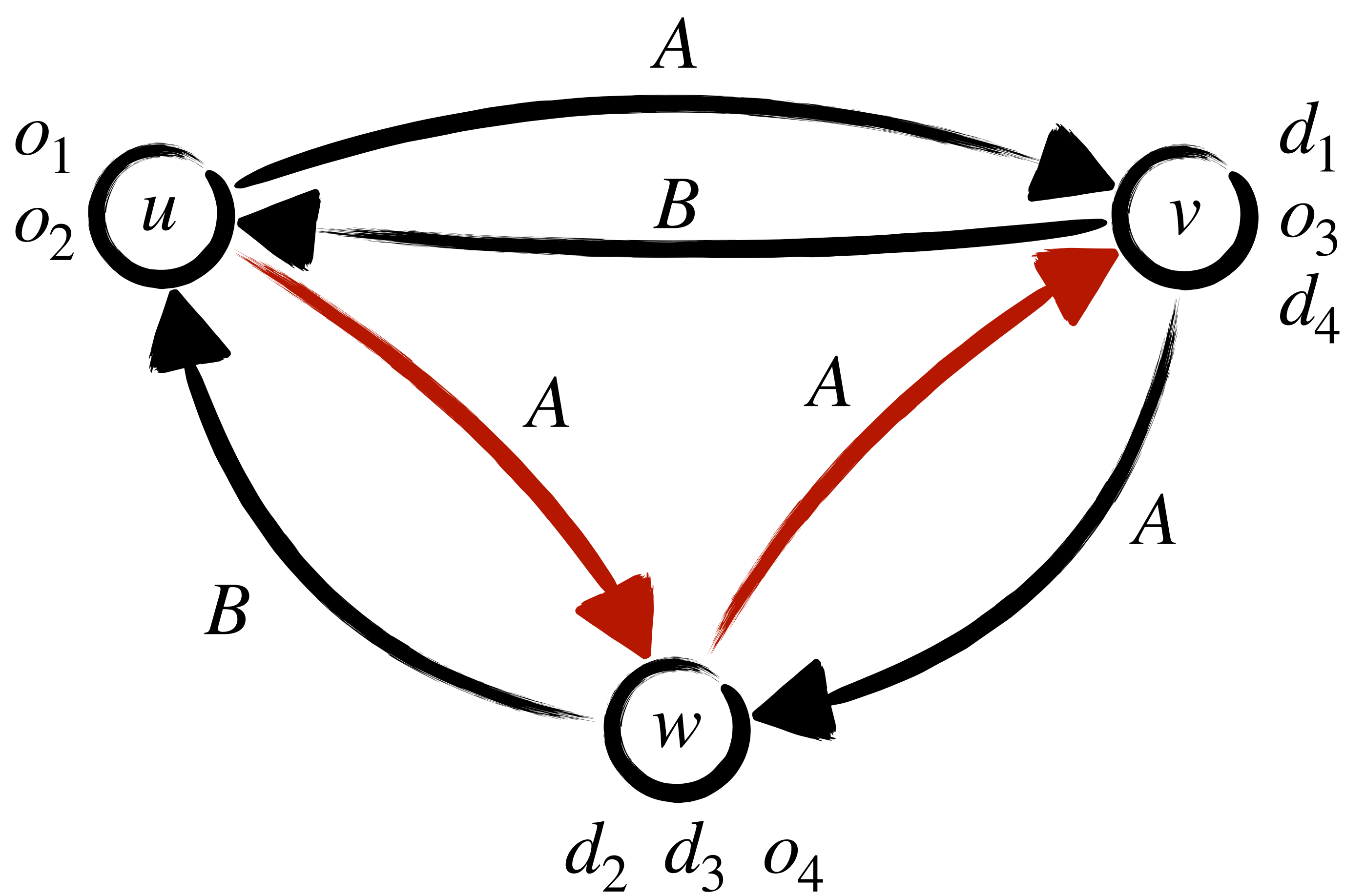
Let s be such that every player takes the two-hop path.

Another network congestion game



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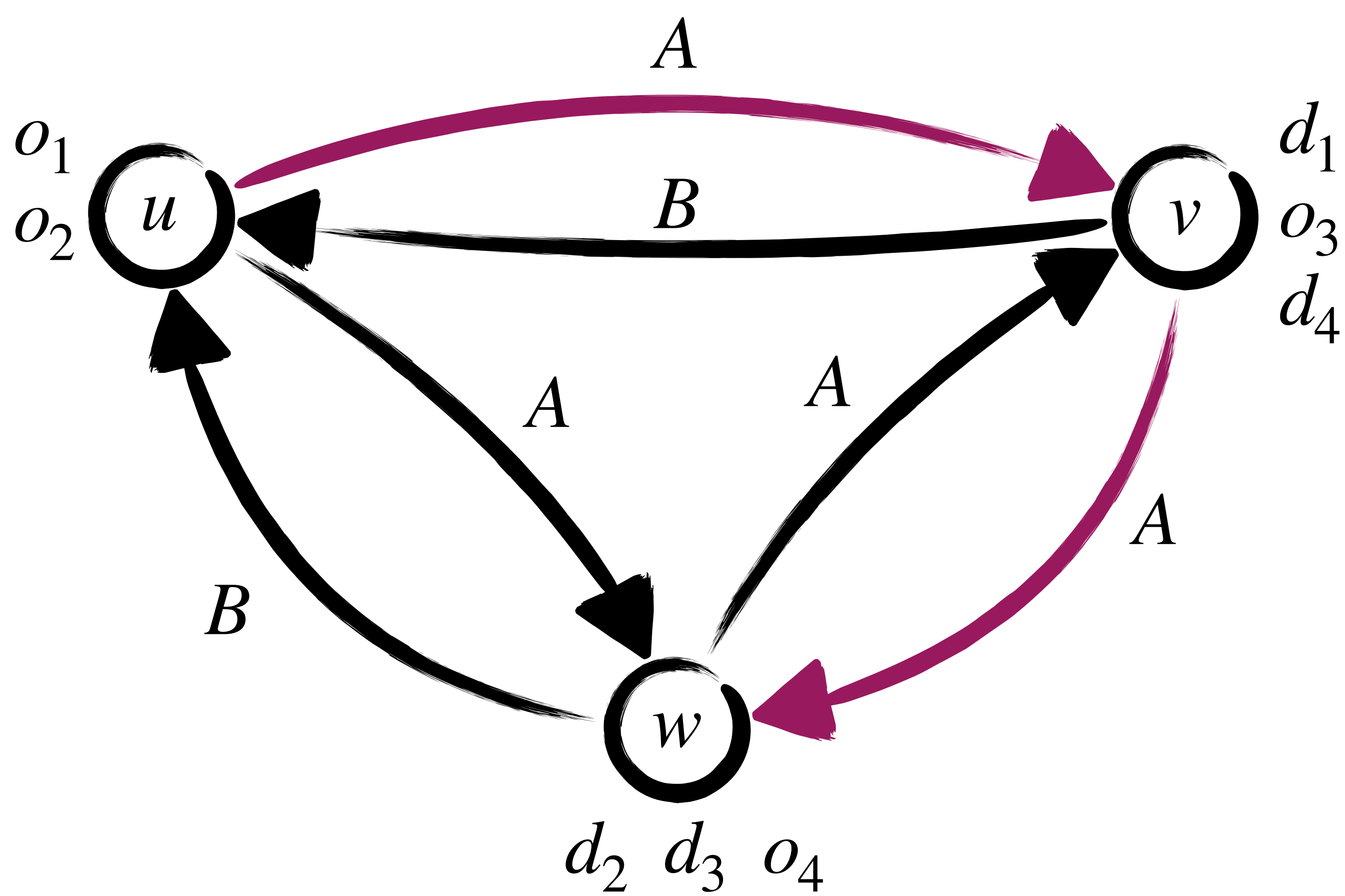

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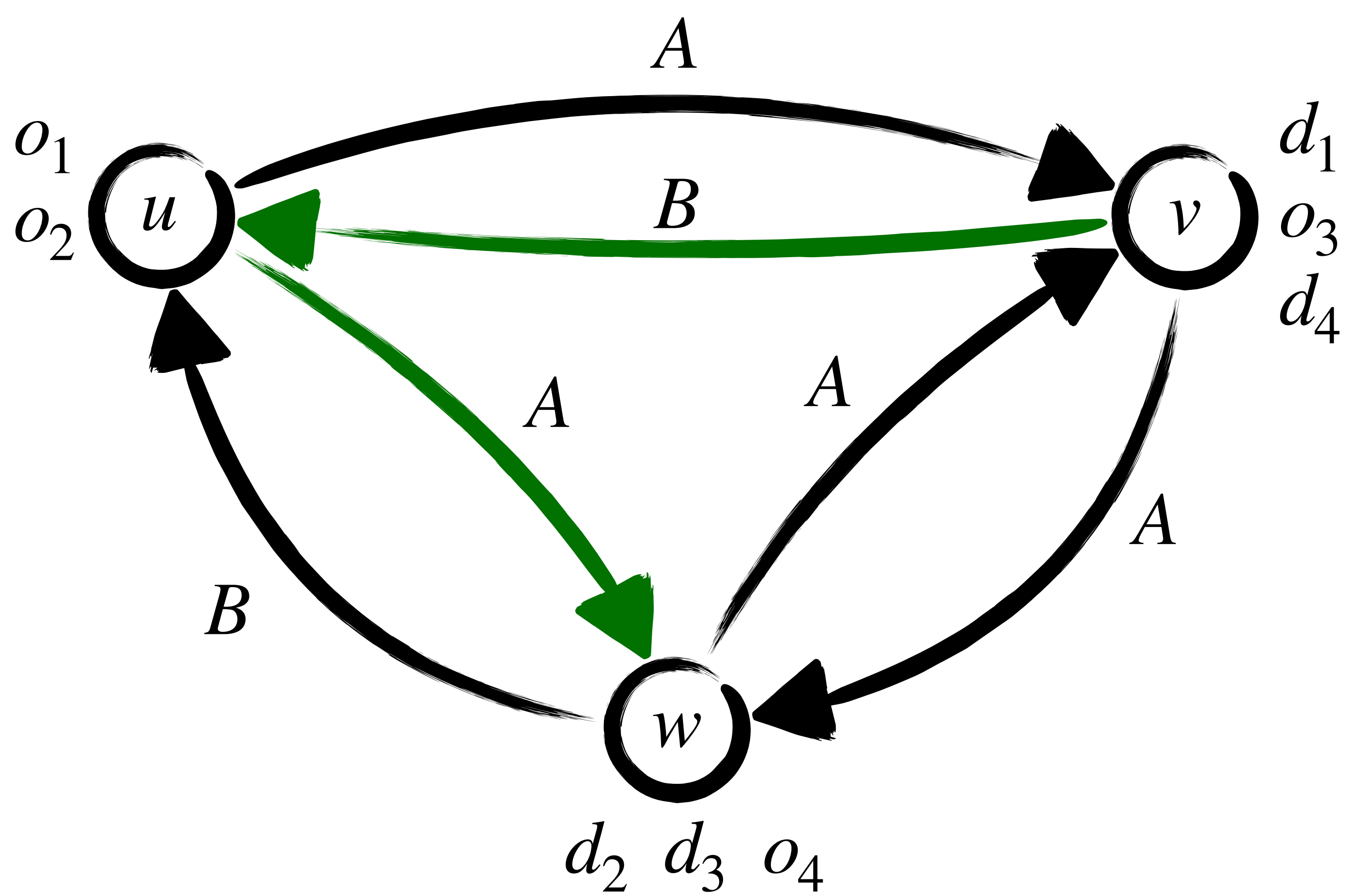

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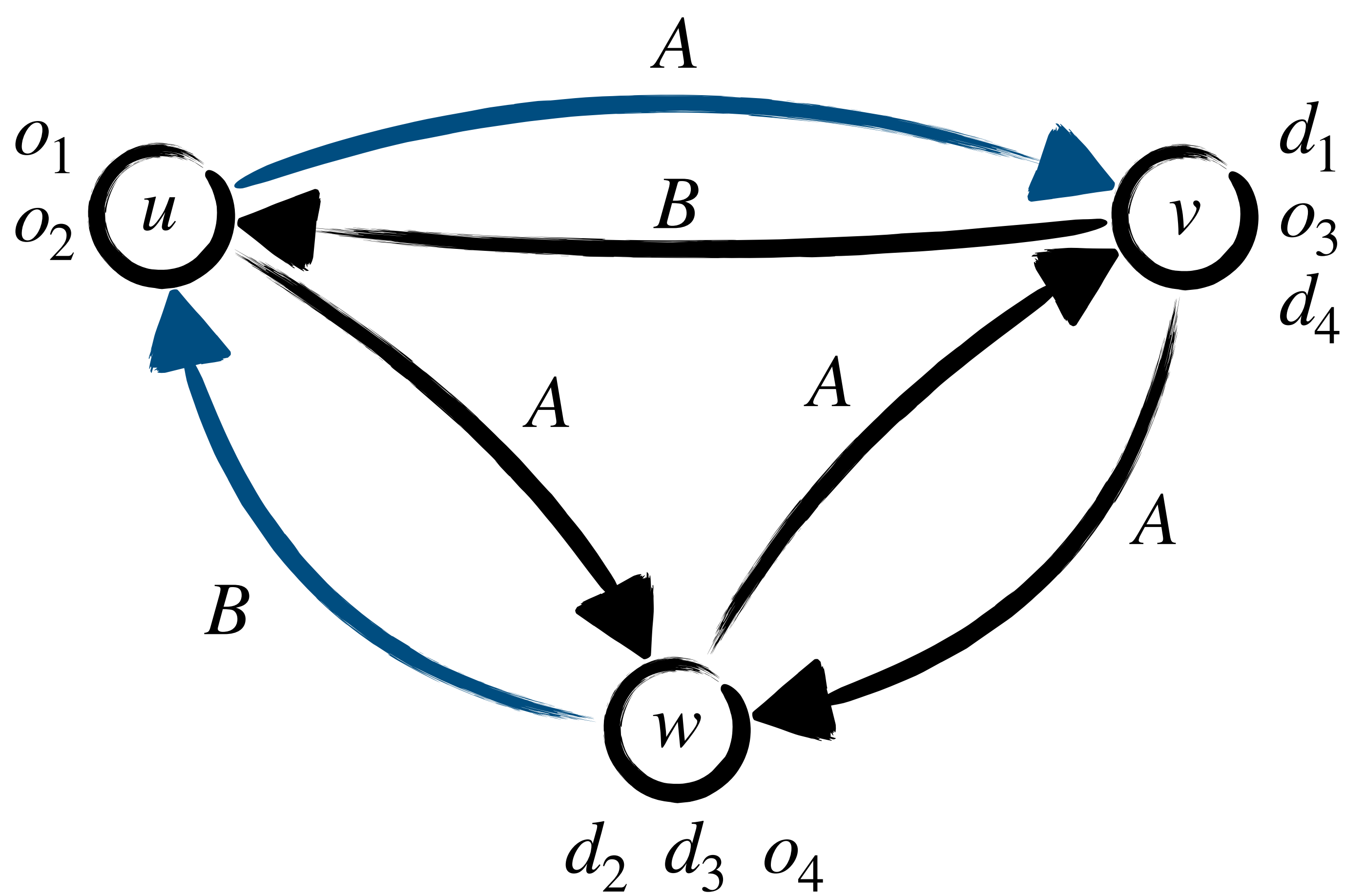

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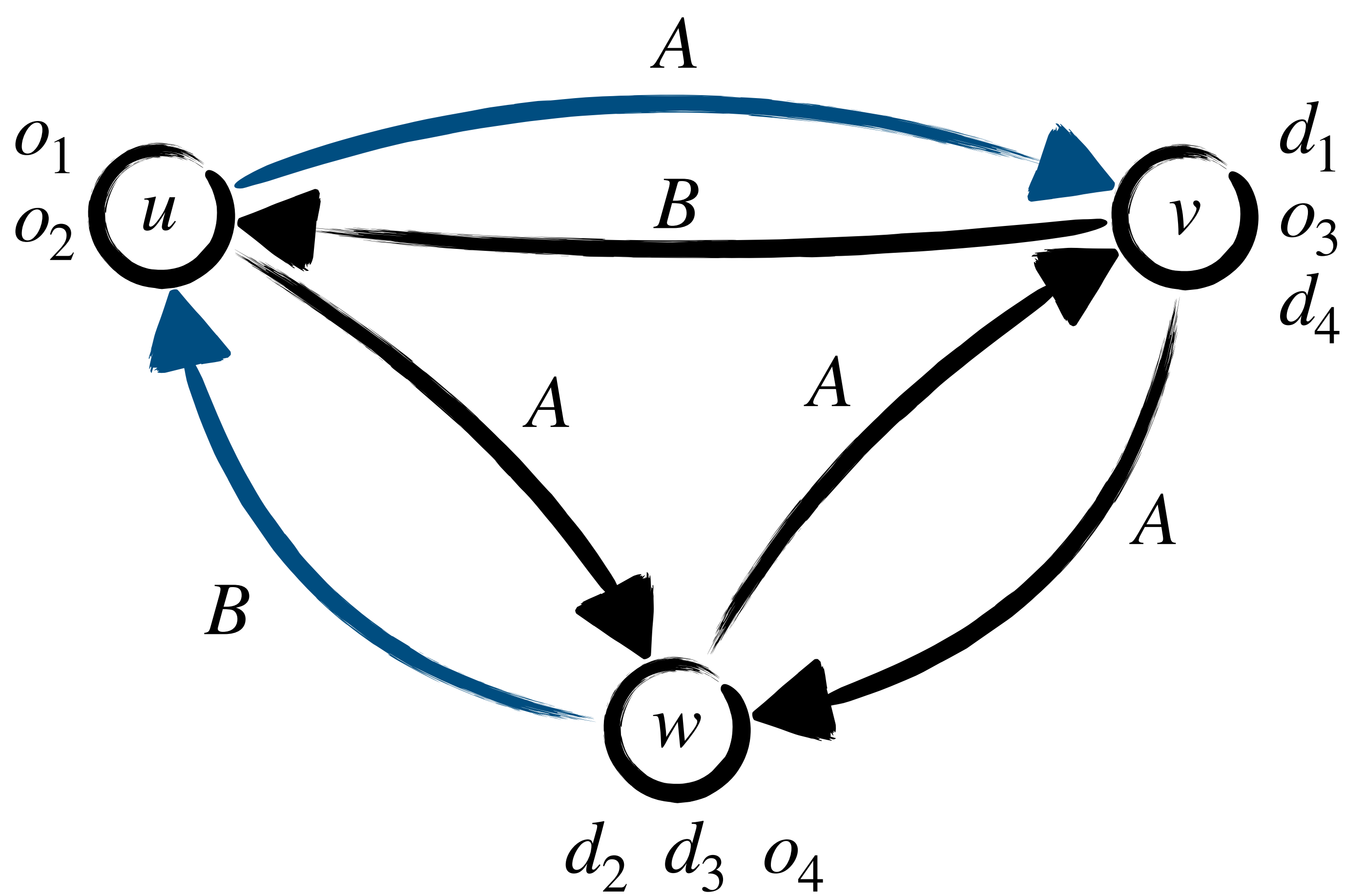

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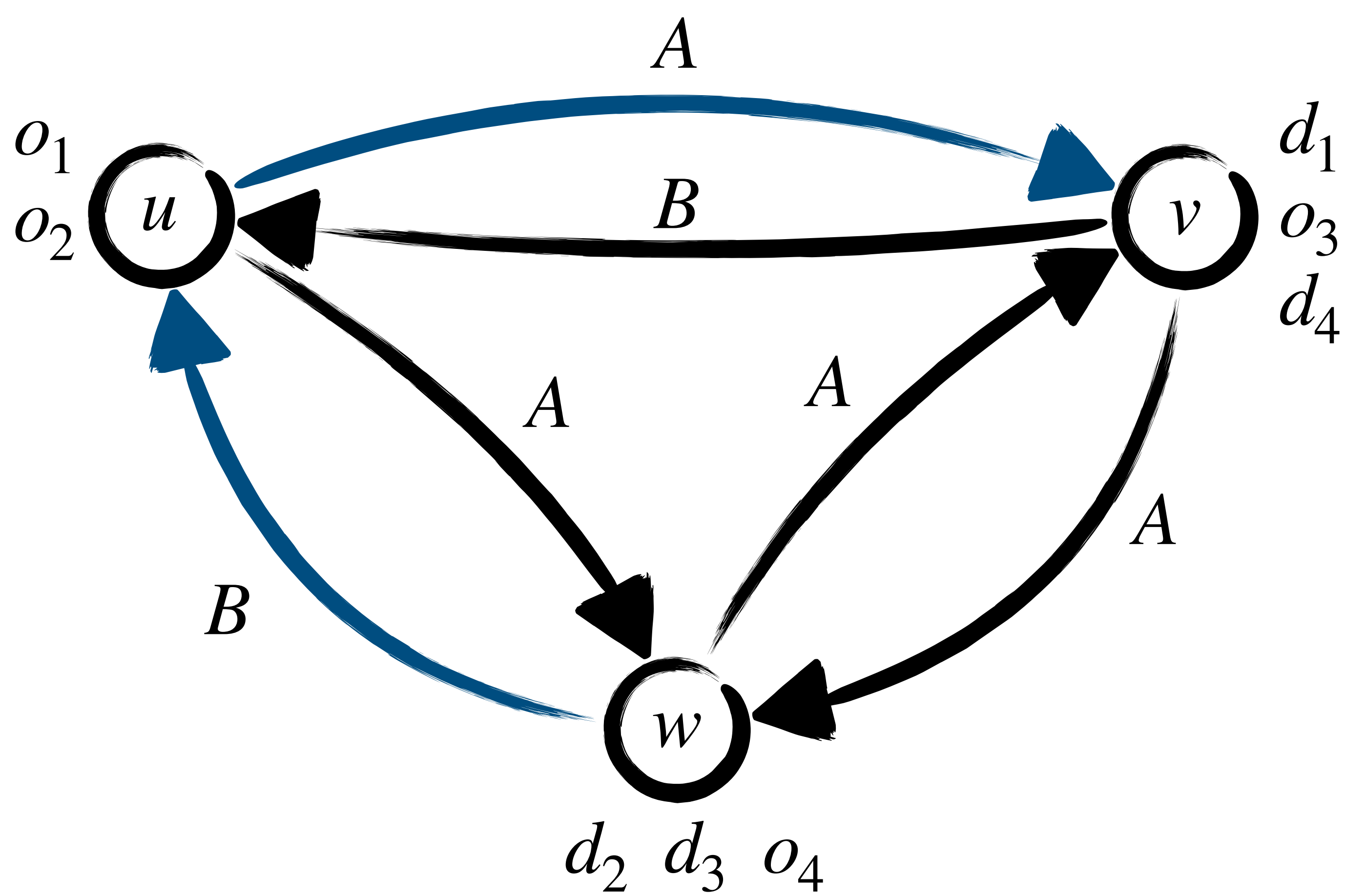

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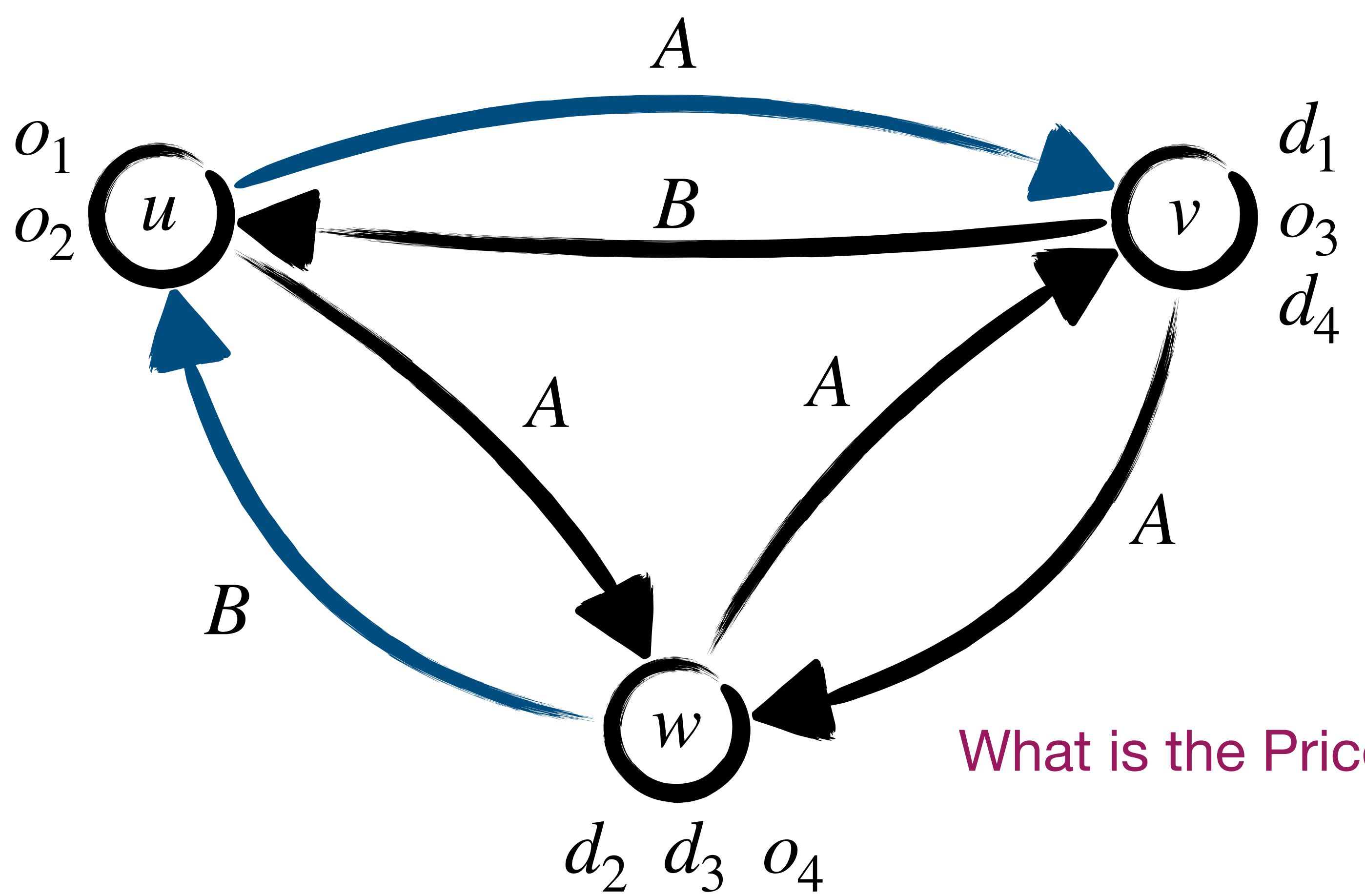
SC(two-hop, two-hop, two-hop) $= 3 + 3 + 2 + 2 = 10$

Another network congestion game



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What is the Price of Anarchy of the game?

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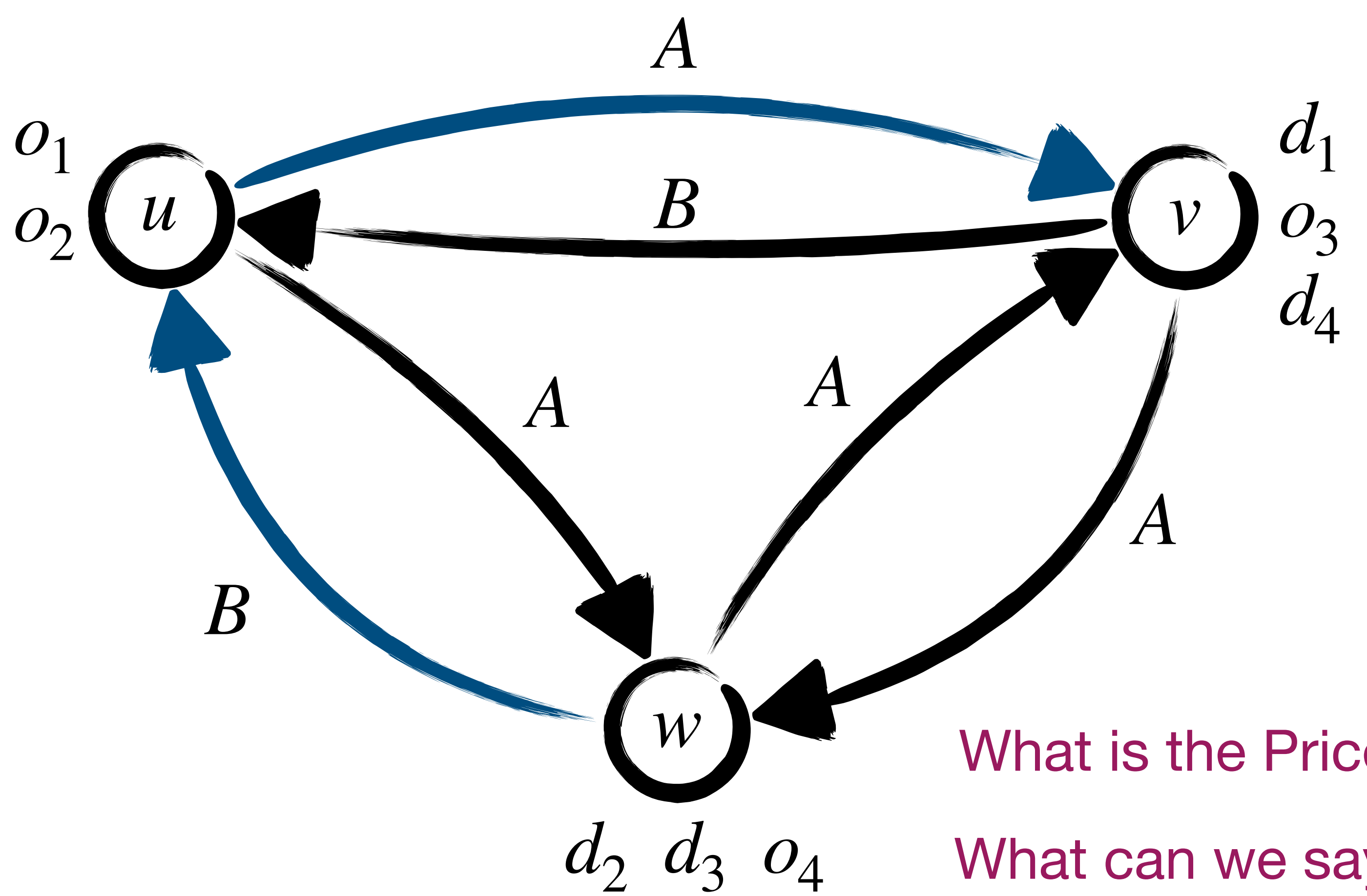

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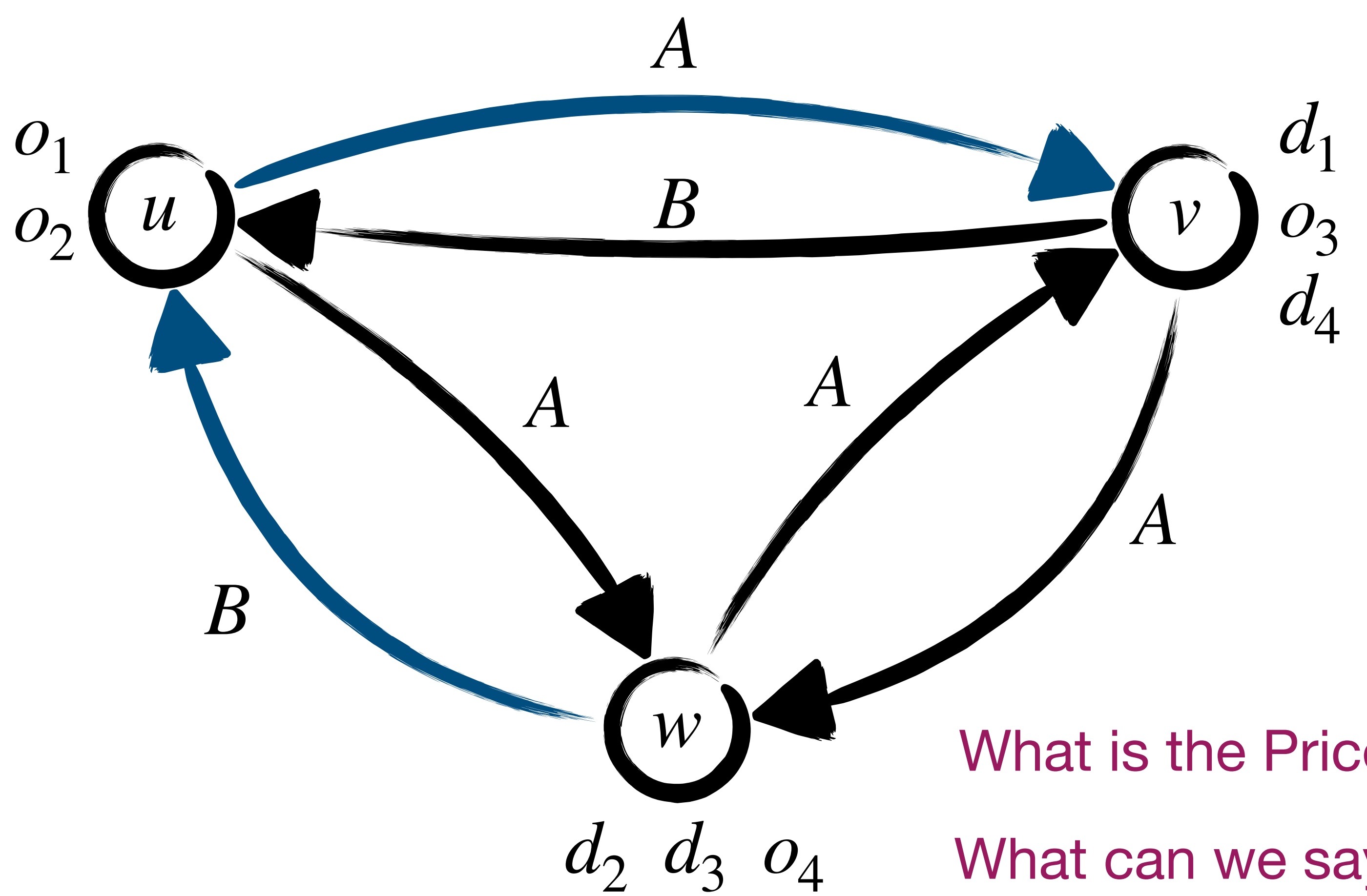

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PoA $(\mathcal{G}_{\text{NC}}) \geq 5/2$

A Price of Anarchy guarantee

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Theorem (Christodoulou and Koutsoupias 2005): The Price of Anarchy of any congestion game with linear cost functions is at most $5/2$.

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why?

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Useful Lemma

Lemma: For all $y, z \in \mathbb{N}$, $y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$

Plugging in, we get:

$$\text{SC}(s) \leq \sum_{r \in R} \left[a_r n_r^*(n_r + 1) + b_r n_r^*\right]$$

$$\leq \sum_{r \in R} \left[a_r \left(\frac{5}{3}(n_r^*)^2 + \frac{1}{3}n_r^2\right) + b_r n_r^*\right]$$

$$= \sum_{r \in R} \left[\frac{5}{3}a_r (n_r^*)^2 + b_r n_r^*\right] + \sum_{r \in R} \frac{1}{3} a_r n_r^2$$

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Lemma (Potential Method): Assume that Φ is a potential function for the game G. If there exist constants $c, d > 0$ such that, for any strategy profile s, it holds that $c \cdot \text{SC}(s) \leq \Phi(s) \leq d \cdot \text{SC}(s)$, then $\text{PoS}(G) \leq \dfrac{d}{c}$.

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One can show that Rosenthal's potential function satisfies the condition with $c = 1/2$ and $d = 1$.

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