Introduction to Algorithms and Data Structures
Lecture 23: Parsing for context-free languages

John Longley

School of Informatics
University of Edinburgh

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The parsing problem

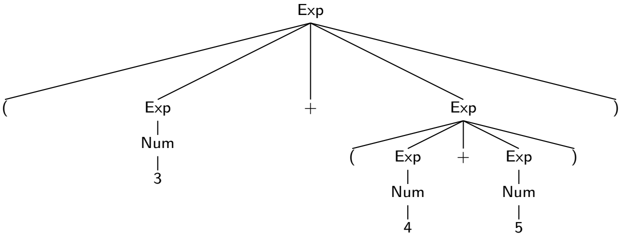
Last time, we saw what a context-free grammar was.

$$\begin{aligned} \text{Exp} &\rightarrow \text{Num} \mid (\text{ Exp } + \text{ Exp }) \\ \text{Num} &\rightarrow 0 \mid \cdots \mid 9 \end{aligned}$$

This time, we'll consider the parsing problem: how do we get from a string of terminals ...

$$(\; 3 + (\; 4 + 5 \;) \;)$$

... to a tree



Often an essential prelude to other tasks (e.g. evaluating an expression!)

The CYK algorithm

We'll describe a general approach that works for *any* CFG, using the Cocke-Younger-Kasami (CYK or CKY) algorithm.
(Seemingly first discovered by Itiroo Sakai in 1961.)
Another example of dynamic programming.

- First see how this algorithm works on a special class of grammars, those in Chomsky normal form (CNF).
- Then see how *any* context-free grammar can be transformed to an 'equivalent' one in CNF.
- CYK parses inputs of length n in time $\Theta(n^3)$. Fine for short sentences, but not practical for long computer programs. Next time, we'll look at parsing algorithms better suited to computer languages: less general, but faster.

What's Chomsky normal form?

Recall that in a CFG, the right-hand side of each production is a (possibly empty) string of terminals and non-terminals. E.g.

$$\text{Exp} \rightarrow (\ \text{Exp} + \text{Exp}\ )$$

A grammar in Chomsky normal form is one in which each RHS consists of

- *either* just two non-terminals (e.g. $X \rightarrow YZ$)
- *or* just one terminal (e.g. $X \rightarrow +$).

We'll see soon what this curious restriction buys us.
Most important point is that RHSs with ≥ 3 symbols are forbidden.

Chomsky normal form: example

The following grammar is in CNF.

Terminals:	book, orange, heavy, my, very
Non-terminals:	NP, Nom, AP, A, Det, Adv
Start symbol:	NP

$$
\begin{array}{rcl}
\text{NP} & \rightarrow & \text{Det Nom} \\
\text{Nom} & \rightarrow & \text{book} \mid \text{orange} \mid \text{AP Nom} \\
\text{AP} & \rightarrow & \text{heavy} \mid \text{orange} \mid \text{Adv A} \\
\text{A} & \rightarrow & \text{heavy} \mid \text{orange} \\
\text{Det} & \rightarrow & \text{my} \\
\text{Adv} & \rightarrow & \text{very}
\end{array}
$$

Generates noun phrases like:

my very heavy orange   my very heavy orange book

(N.B. CNF grammars often involve some duplication!
Writing AP $\rightarrow$ A would be simpler, but not CNF.)

CYK parsing: the idea

Let's insert 'position markers' in the input string we wish to parse:

$_0$ my $_1$ very $_2$ heavy $_3$ orange $_4$ book $_5$

We can then talk about substrings of the input: e.g. the pair (2,4) indicates the substring 'heavy orange'.

Primary question: Can the entire string (0,5) be derived from the start symbol NP? If so, how?

As is common in Dynamic Programming, we approach this by generalizing our objective slightly: Which substrings can be derived from which non-terminals?

We store the solutions to these 'subproblems' in a 2-dim array: entry for (i, j) (where $i < j$) records possible analyses of the substring indicated by (i, j).

Broadly speaking, we work our way from shorter to longer substrings (some flexibility re precise ordering of subproblems).

Filling out the CYK chart: example

$$\begin{array}{rcl} NP & \rightarrow & Det\ Nom \\ Nom & \rightarrow & book \mid orange \mid AP\ Nom \\ AP & \rightarrow & heavy \mid orange \mid Adv\ A \end{array} \qquad \begin{array}{rcl} A & \rightarrow & heavy \mid orange \\ Det & \rightarrow & my \\ Adv & \rightarrow & very \end{array}$$

$_0$ my $_1$ very $_2$ heavy $_3$ orange $_4$ book $_5$

i \ j	1 my	2 very	3 heavy	4 orange	5 book
0 my	Det			NP	NP
1 very		Adv	AP	Nom	Nom
2 heavy			A,AP	Nom	Nom
3 orange				Nom,A,AP	Nom
4 book					Nom

CYK: The general algorithm

```
CYK (s,G):                  # s=input string, G=CNF grammar
    n = length(s)
    allocate table[0,...,n−1][1,...,n]
    for j = 1 to n          # columns
        for (X → t) ∈ G
            if t = s[j−1]
                add X to table[j−1,j]     # diagonal cell
        for i = j−2 downto 0              # rows
            for k = i+1 to j−1            # possible splits
                for (X → YZ) ∈ G
                    if Y ∈ table[i,k] and Z ∈ table [k,j]
                        add X to table[i,j]     # non-diagonal cell
    return table
```

From recognizer to parser

- So far, we just have a recognizer: a way of determining whether a string belongs to the given language.
- Changing this to a parser requires recording which existing constituents were combined to make each new constituent.

$_0$ a $_1$ very $_2$ heavy $_3$ orange $_4$ book $_5$

		1 *a*	2 *very*	3 *heavy*	4 *orange*	5 *book*
0	a	Det			NP	NP
1	very		Adv	AP	Nom	Nom
2	heavy			A,AP	Nom	Nom
3	orange				Nom,A,AP	Nom
4	book					Nom

- The algorithm identifies all possible parses.
  There may also be phantom constituents that don't form part of any complete syntax tree (e.g. 'my very heavy orange').

Runtime of CYK

Looking at the pseudocode for CYK, we have three nested for-loops, each of which we go round $\leq n$ times.
And within them, some iteration over the grammar rules.

So for any fixed grammar G, the algorithm runs in time $O(n^3)$.
(If we allow grammar to vary, runtime is $O(mn^3)$, where m is 'size' of grammar.)

What would happen if we allowed ternary rules, e.g. $A \rightarrow BCD$?

To fill a cell (i, j), we'd need to consider all possible three-way splits $(i, k), (k, l), (l, j)$ where $i < k < l < j$.
Number of these is quadratic in $j - i$.
So our overall runtime would go up to $\Theta(n^4)$.

That's the main reason we like Chomsky normal form (there are other minor benefits).

More on Chomsky normal form

Recall: a context-free grammar $\mathcal{G} = (\Sigma, N, S, P)$ is in Chomsky normal form (CNF) if all productions are of the form

$$A \rightarrow BC \quad \text{or} \quad A \rightarrow a \qquad (A, B, C \in N,\ a \in \Sigma)$$

Theorem: Disregarding the empty string, every CFG $\mathcal{G}$ is equivalent to a grammar $\mathcal{G}'$ in Chomsky normal form. $(\mathcal{L}(\mathcal{G}') = \mathcal{L}(\mathcal{G}) - \{\epsilon\})$
And there's an algorithm which, given $\mathcal{G}$, finds a suitable $\mathcal{G}'$.

Key idea: To eliminate rules with ≥ 3 symbols on the RHS, we could replace e.g.

$$X \rightarrow ABCD \quad \text{by} \quad X \rightarrow AY,\ Y \rightarrow BZ,\ Z \rightarrow CD$$

where Y, Z are newly added nonterminals.

Converting to Chomsky Normal Form

Consider for example the grammar

$$S \rightarrow TT \mid [S] \qquad T \rightarrow \epsilon \mid (T)$$

Step 1: Apply trick on last slide to rules with ≥ 3 symbols on RHS. In this case, apply it to $S \rightarrow [S]$ and $T \rightarrow (T)$:

$$S \rightarrow TT \mid [W \qquad T \rightarrow \epsilon \mid (V$$

$$W \rightarrow S] \qquad V \rightarrow T)$$

Step 2: Identify the set E of all non-terminals X such that ϵ can be derived from X (nullable non-terminals).

In this case, $T \rightarrow \epsilon$ tells us $T \in E$. Then $S \rightarrow TT$ tells us $S \in E$. And that's all. So $E = \{S, T\}$.

In general, E is the smallest set such that if $X \rightarrow Y_1 \dots Y_r \in P$ and $Y_1, \dots, Y_r \in E$ then $X \in E$ (allowing $r = 0$ here).

Converting to Chomsky Normal Form, ctd.

$$S \rightarrow TT \mid [W \qquad T \rightarrow \epsilon \mid (V$$
$$W \rightarrow S] \qquad V \rightarrow T)$$

Step 3: Delete all ϵ-productions.

To compensate, for each rule $X \rightarrow Y\alpha$ or $X \rightarrow \alpha Y$, where $Y \in E$ and $\alpha \neq \epsilon$, add a new rule $X \rightarrow \alpha$.

In this case, since $E = \{S, T\}$, we get:

$$S \rightarrow TT \mid T \mid [W \qquad T \rightarrow (V$$
$$W \rightarrow S] \mid] \qquad V \rightarrow T) \mid)$$

Step 4: Remove unit productions $X \rightarrow Y$.

To compensate, for every rule $Y \rightarrow \alpha$, add in $X \rightarrow \alpha$.

In this case, do this for $S \rightarrow T$:

$$S \rightarrow TT \mid (V \mid [W \qquad T \rightarrow (V$$
$$W \rightarrow S] \mid] \qquad V \rightarrow T) \mid)$$

Converting to Chomsky Normal Form, ctd., ctd.

$$S \rightarrow TT \mid (V \mid [W \qquad T \rightarrow (V$$
$$W \rightarrow S] \mid] \qquad V \rightarrow T) \mid)$$

By this stage, all RHSs consist of 1 terminal or 2 symbols. So just need to get rid of terminals from the 'binary' rules.

Step 5: For each terminal a, add a fresh nonterminal Z_a and a production $Z_a \rightarrow a$, then replace a by Z_a in all binary rules.

In this case, we add four rules:

$$Z_(\rightarrow (\qquad Z_) \rightarrow) \qquad Z_[\rightarrow [\qquad Z_] \rightarrow]$$

And rewrite the existing rules to:

$$S \rightarrow TT \mid Z_(V \mid Z_[W \qquad T \rightarrow Z_(V$$
$$W \rightarrow SZ_] \mid] \qquad V \rightarrow TZ_) \mid)$$

The grammar is now in Chomsky Normal Form, and we're done.

Assorted remarks

- Given a CFG $\mathcal{G}$, we can do the above (once for all) to convert it to a CNF grammar $\mathcal{G}'$, then run CYK for $\mathcal{G}'$ (many times).
- This will give us a syntax tree w.r.t. $\mathcal{G}'$. Bit of work to translate back to a tree w.r.t. $\mathcal{G}$ — not very hard/interesting.
- If $\mathcal{G}$ has m rules, our algorithm gives a $\mathcal{G}'$ with $O(m^2)$ rules. Quadratic blow-up possible, but not a problem in practice.
- Versions of CYK are quite widely used in Natural Language context (where sentences typically have < 100 words). But $\Theta(n^3)$ parsing not good enough for computer languages.

Reading

Recommended: D. Jurafsky and J.H. Martin,
Speech and Language Processing, 3rd ed. (draft).
Chapter 13 (Constituency parsing), Sections 1 and 2.
Available at `https://web.stanford.edu/~jurafsky/slp3`