Algorithmic Game Theory and Applications

The Simplex Method

Linear Programs in Standard Form

$$\text{maximise} \qquad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1\in}^{n} \alpha_{ij} x_j \leq b_i, \quad i = 1, ..., m$$

$$x_j \geq 0, \quad j = 1, ..., n$$

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Return an optimal solution (i.e., a feasible solution that maximises the objective function), or

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Return that the LP is unbounded.

The Simplex Method (explained via example)

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &\leq 5 \\ 4x_1 + x_2 + 2x + 3 &\leq 11 \\ 3x_1 + 4x_2 + 2x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

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e.g., for the constraint $2x_1 + 3x_2 + x_3 \leq 5$, we introduce variable w_1 and we write

$$w_1 = 5 - 2x_1 - 3x_2 - x_3$$

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subject to

$$2x_1 + 3x_2 + x_3 \leq 5$$
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Maximise $5x_1 + 4x_2 + 3x_3$

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$$w_1 = 5 - 2x_1 + 3x_2 + x_3$$

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w_1 &= 5 - 2x_1 + 3x_2 + x_3 \\
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& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Is this equivalent to the original LP?

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e.g., for the constraint $2x_1 + 3x_2 + x_3 \leq 5$, we introduce variable w_1 and we write

$$w_1 = 5 - 2x_1 - 3x_2 - x_3$$

We also introduce a slack variable ζ for the objective function.

Step 1: Slack Variables

Maximise $\zeta = 5x_1 + 4x_2 + 3x_3$

subject to

$$w_1 = 5 - 2x_1 + 3x_2 + x_3$$

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$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

Dictionaries

Maximise $\zeta = \quad +5\ x_1 \quad +4\ x_2 \quad +3\ x_3$

subject to

$$\begin{aligned} w_1 &= 5 \quad -2\ x_1 \quad -3\ x_2 \quad -\ x_3 \\ w_2 &= 11 \quad -4\ x_1 \quad -\ x_2 \quad -2\ x_3 \\ w_3 &= 8 \quad -3\ x_1 \quad -4\ x_2 \quad -2\ x_3 \end{aligned}$$

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basic variables

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$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

nonbasic variables

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Maximise $5x_1 + 4x_2 + 3x_3$

subject to $w_1 = 5 - 2x_1 + 3x_2 + x_3$

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Improve this solution to some $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3$ such that $5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 > 5x_1 + 4x_2 + 3x_3$

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Continue until no further improvement is possible (in that case we are at an optimal solution).

Dictionaries

$$
\begin{array}{lrrrrr}
\textbf{Maximise} & \zeta = & & +5\ x_1 & +4\ x_2 & +3\ x_3 \\
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$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

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basic variables

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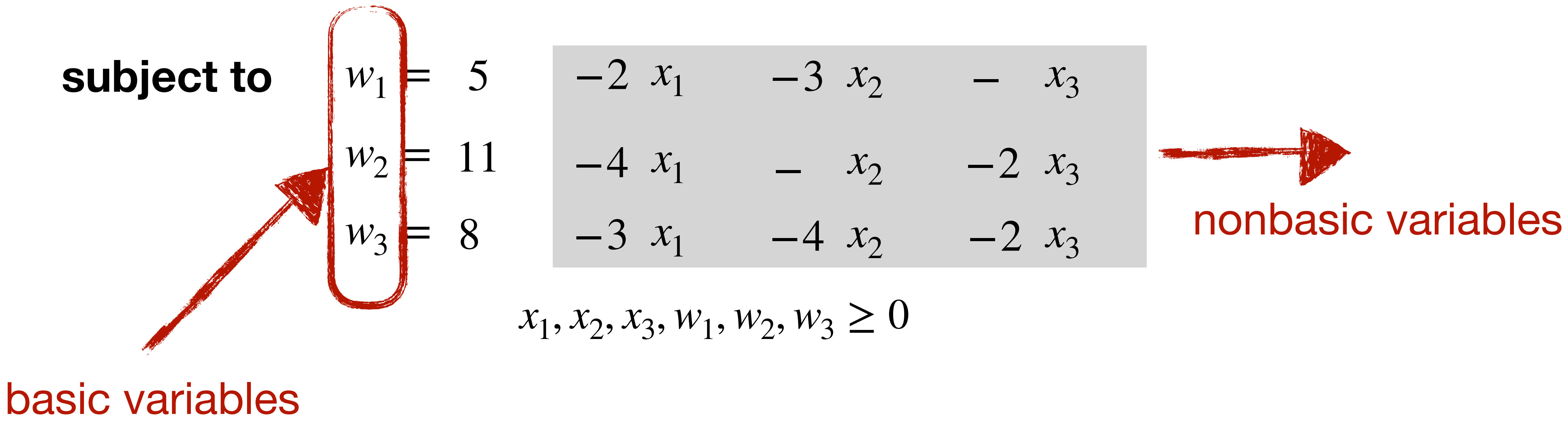
Maximise $\zeta = \quad +5\ x_1 \quad +4\ x_2 \quad +3\ x_3$



subject to

$$\begin{array}{rcrrrr} w_1 &=& 5 & -2\ x_1 & -3\ x_2 & -\ x_3 \\ w_2 &=& 11 & -4\ x_1 & -\ x_2 & -2\ x_3 \\ w_3 &=& 8 & -3\ x_1 & -4\ x_2 & -2\ x_3 \end{array}$$

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Start with a feasible solution $x_1, x_2, x_3, w_1, w_2, w_3$

Suggestions?

Dictionaries

Maximise $\zeta = \quad +5x_1 \quad +4x_2 \quad +3x_3$

subject to

$$\begin{array}{rcrrrr} w_1 & = & 5 & -2x_1 & -3x_2 & -\ x_3 \\ w_2 & = & 11 & -4x_1 & -\ x_2 & -2x_3 \\ w_3 & = & 8 & -3x_1 & -4x_2 & -2x_3 \end{array}$$

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$$x_1 = x_2 = x_3 = 0$$

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$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

nonbasic variables

basic variables

A solution obtained by setting all the nonbasic variables to 0 is called a basic feasible solution.

$x_1 = x_2 = x_3 = 0 \qquad\qquad w_1 = 5, w_2 = 11, w_3 = 8$

Step 2: Improving the solution

Maximise $\zeta = \quad +5x_1 \quad +4x_2 \quad +3x_3$

subject to

$$\begin{aligned} w_1 &= 5 && -2x_1 && -3x_2 && -x_3 \\ w_2 &= 11 && -4x_1 && -x_2 && -2x_3 \\ w_3 &= 8 && -3x_1 && -4x_2 && -2x_3 \end{aligned}$$

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We can increase the value of some nonbasic variable, e.g., x_1

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We should not violate any constraints though!

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We should not violate any constraints though!

We don't want any of the slack variables to become negative.

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\begin{aligned}
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\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

For w_1, x_1 can become as large as $5/2 = 30/12$.

Step 2: Improving the solution

Maximise $\zeta = \quad +5x_1 + 4x_2 + 3x_3$

subject to

$$\begin{aligned} w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \end{aligned}$$

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$$x_1 = 5/2, \;\; x_2 = x_3 = 0 \qquad w_1 = 0, \, w_2 = 1, \, w_3 = 1/2$$

For w_1, x_1 can become as large as $5/2 = 30/12$.

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We need to rearrange the inequalities, so that x_1 now only appears on the left.

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We need to rearrange the inequalities, so that x_1 now only appears on the left.

This gives rise to a new dictionary, where x_1 is now basic and w_1 is nonbasic.

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$$

entering variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

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entering variable

leaving variable

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entering variable

$\}$ **just rearranging**

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entering variable

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just rearranging

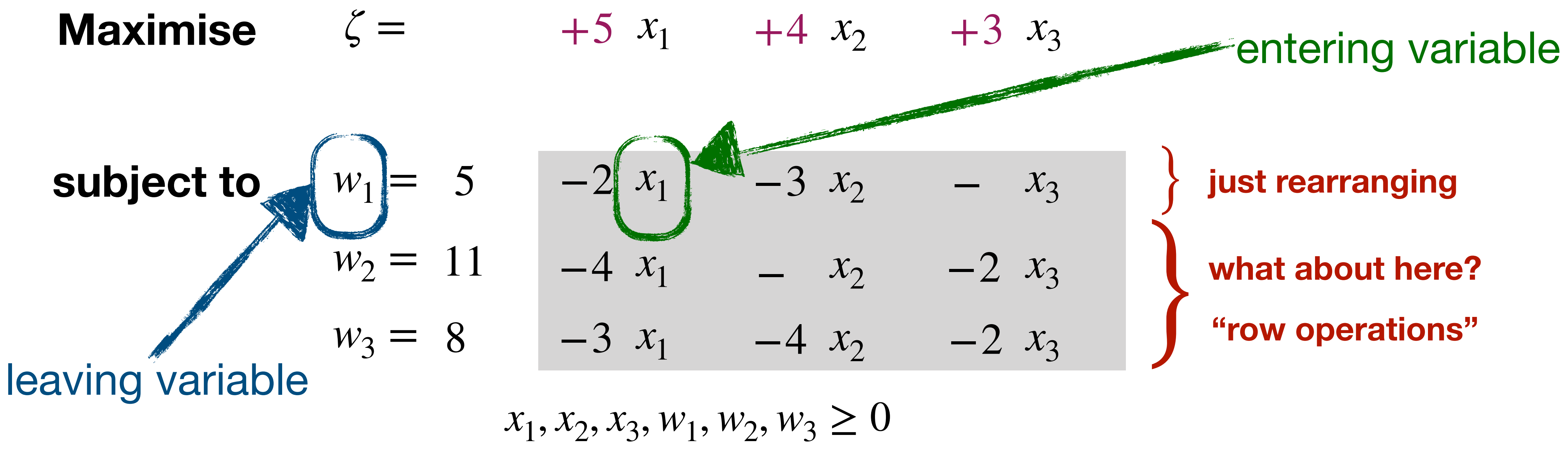
what about here?

$$x_1 = 5/2, \quad x_2 = x_3 = 0 \qquad w_1 = 0,\ w_2 = 1,\ w_3 = 1/2$$

We need to rearrange the inequalities, so that x_1 now only appears on the left.

This gives rise to a new dictionary, where x_1 is now basic and w_1 is nonbasic.

Step 2: Improving the solution



$$\begin{array}{llrrrr} \textbf{Maximise} & \zeta = & & +5\,x_1 & +4\,x_2 & +3\,x_3 \\ \textbf{subject to} & \boxed{w_1} = & 5 & -2\,\boxed{x_1} & -3\,x_2 & -\,x_3 \\ & w_2 = & 11 & -4\,x_1 & -\,x_2 & -2\,x_3 \\ & w_3 = & 8 & -3\,x_1 & -4\,x_2 & -2\,x_3 \end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

entering variable

leaving variable

just rearranging

what about here?

“row operations”

$$x_1 = 5/2, \quad x_2 = x_3 = 0 \qquad w_1 = 0, \; w_2 = 1, \; w_3 = 1/2$$

We need to rearrange the inequalities, so that x_1 now only appears on the left.

This gives rise to a new dictionary, where x_1 is now basic and w_1 is nonbasic.

Step 2: Improving the solution

Maximise $\zeta = \quad +5\ x_1 \quad +4\ x_2 \quad +3\ x_3$

subject to

$$\begin{array}{rcrrrrrr} w_1 &=& 5 & -2 & x_1 & -3 & x_2 & - & x_3 \\ w_2 &=& 11 & -4 & x_1 & - & x_2 & -2 & x_3 \\ w_3 &=& 8 & -3 & x_1 & -4 & x_2 & -2 & x_3 \end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

just rearranging

what about here?

“row operations”

Step 2: Improving the solution

Maximise $\zeta = +5\,x_1 \quad +4\,x_2 \quad +3\,x_3$

subject to

$$\left.w_1 = 5 \quad -2\,x_1 \quad -3\,x_2 \quad -\,x_3\right\}\ \textbf{just rearranging}$$

$$\left.\begin{array}{l} w_2 = 11 \quad -4\,x_1 \quad -\,x_2 \quad -2\,x_3 \\ w_3 = 8 \quad -3\,x_1 \quad -4\,x_2 \quad -2\,x_3 \end{array}\right\}\ \begin{array}{l}\textbf{what about here?} \\ \textbf{“row operations”}\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

Notice that $w_2 - 2w_1 = 11 - 4x_1 - x_2 - 2x_3 - 10 + 4x_1 + 6x_2 + 2x_3$

Step 2: Improving the solution

Maximise $\zeta = \quad +5\ x_1 \quad +4\ x_2 \quad +3\ x_3$

subject to

$$\begin{array}{rcrrrr} w_1 & = & 5 & -2\ x_1 & -3\ x_2 & -\ \ x_3 \\ w_2 & = & 11 & -4\ x_1 & -\ \ x_2 & -2\ x_3 \\ w_3 & = & 8 & -3\ x_1 & -4\ x_2 & -2\ x_3 \end{array}$$

just rearranging

what about here?

“row operations”

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

Notice that $w_2 - 2w_1 = 11 - 4x_1 - x_2 - 2x_3 - 10 + 4x_1 + 6x_2 + 2x_3$

$$\Rightarrow w_2 = 1 + 2w_1 + 5x_2$$

Step 2: Improving the solution

Maximise $\zeta = +5x_1 +4x_2 +3x_3$

subject to

$$\begin{array}{rcrrrrl} w_1 &=& 5 & -2\,x_1 & -3\,x_2 & -\,x_3 & \left.\right\}\ \text{just rearranging} \\ w_2 &=& 11 & -4\,x_1 & -\,x_2 & -2\,x_3 & \left.\begin{array}{l}\text{what about here?} \\ \text{“row operations”}\end{array}\right. \\ w_3 &=& 8 & -3\,x_1 & -4\,x_2 & -2\,x_3 & \end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

Notice that $w_2 - 2w_1 = 11 - 4x_1 - x_2 - 2x_3 - 10 + 4x_1 + 6x_2 + 2x_3$

$\Rightarrow w_2 = 1 + 2w_1 + 5x_2$ x_1 has been eliminated

The New Dictionary

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\
w_2 &= 1 + 2\, w_1 + 5\, x_2 \\
w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

The New Dictionary

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 & -0.5\, w_1 & \quad -1.5\, x_2 & \quad -0.5\, x_3 \\
w_2 &= 1 & +2\, w_1 & \quad +5\, x_2 & \\
w_3 &= 0.5 & +1.5\, w_1 & \quad +0.5\, x_2 & \quad -0.5\, x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

basic variables

The New Dictionary

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$
\begin{array}{rcl|lll}
x_1 & = & 2.5 & -0.5\, w_1 & -1.5\, x_2 & -0.5\, x_3 \\
w_2 & = & 1 & +2\, w_1 & +5\, x_2 & \\
w_3 & = & 0.5 & +1.5\, w_1 & +0.5\, x_2 & -0.5\, x_3
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

nonbasic variables

basic variables

The New Dictionary

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5\,w_1 - 1.5\,x_2 - 0.5\,x_3 \\ w_2 &= 1 + 2\,w_1 + 5\,x_2 \\ w_3 &= 0.5 + 1.5\,w_1 + 0.5\,x_2 - 0.5\,x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

nonbasic variables

basic variables

$$w_1 = 0,\ x_2 = 0\ x_3 = 0$$

The New Dictionary

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$\begin{array}{rcllll}
x_1 & = & 2.5 & -0.5\,w_1 & -1.5\,x_2 & -0.5\,x_3 \\
w_2 & = & 1 & +2\,w_1 & +5\,x_2 & \\
w_3 & = & 0.5 & +1.5\,w_1 & +0.5\,x_2 & -0.5\,x_3
\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

nonbasic variables

basic variables

$$w_1 = 0,\ x_2 = 0\ x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

The New Dictionary

Maximise $\underline{\zeta = 12.5} - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

the objective function value has increased

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\ w_2 &= 1 + 2\, w_1 + 5\, x_2 \\ w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3 \end{aligned}$$

nonbasic variables

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

basic variables

$$w_1 = 0,\ x_2 = 0\ x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

The New Dictionary

Maximise $\underline{\zeta = \quad 12.5} \; -2.5\, w_1 \quad -3.5\, x_2 \quad +0.5\, x_3$

the objective function value has increased

subject to

$$
\begin{array}{rcl|llll}
x_1 & = & 2.5 & -0.5\, w_1 & -1.5\, x_2 & -0.5\, x_3 \\
w_2 & = & 1 & +2\, w_1 & +5\, x_2 & \\
w_3 & = & 0.5 & +1.5\, w_1 & +0.5\, x_2 & -0.5\, x_3
\end{array}
$$

nonbasic variables

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

basic variables

$$w_1 = 0,\ x_2 = 0\ x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

Which variable should we try to increase next?

Step 3: Improving the solution

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$
\begin{array}{rcl}
x_1 &=& 2.5 - 0.5\,w_1 - 1.5\,x_2 - 0.5\,x_3 \\
w_2 &=& 1 + 2\,w_1 + 5\,x_2 \\
w_3 &=& 0.5 + 1.5\,w_1 + 0.5\,x_2 - 0.5\,x_3
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

Step 3: Improving the solution

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$\begin{array}{rcl}
x_1 & = & 2.5 - 0.5 w_1 - 1.5\, x_2 - 0.5\, x_3 \\
w_2 & = & 1 + 2\, w_1 + 5\, x_2 \\
w_3 & = & 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3
\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

For x_1, x_3 can become as large as $2.5/0.5 = 5$.

Step 3: Improving the solution

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5 w_1 - 1.5\, x_2 - 0.5\, x_3 \\ w_2 &= 1 + 2\, w_1 + 5\, x_2 \\ w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

For x_1, x_3 can become as large as $2.5/0.5 = 5$.

For w_2, x_3 can become as large as ∞.

Step 3: Improving the solution

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5w_1 - 1.5\,x_2 - 0.5\,x_3 \\ w_2 &= 1 + 2\,w_1 + 5\,x_2 \\ w_3 &= 0.5 + 1.5\,w_1 + 0.5\,x_2 - 0.5\,x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

For x_1, x_3 can become as large as $2.5/0.5 = 5$.

For w_2, x_3 can become as large as ∞.

For w_3, x_3 can become as large as $0.5/0.5 = 1$.

Step 3: Improving the solution

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5\,w_1 - 1.5\,x_2 - 0.5\,x_3 \\ w_2 &= 1 + 2\,w_1 + 5\,x_2 \\ w_3 &= 0.5 + 1.5\,w_1 + 0.5\,x_2 - 0.5\,x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_3 = 1, \quad w_1 = x_2 = 0$$

For x_1, x_3 can become as large as $2.5/0.5 = 5$.

For w_2, x_3 can become as large as ∞.

For w_3, x_3 can become as large as $0.5/0.5 = 1$.

Step 3: Improving the solution

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 - 0.5\,w_1 - 1.5\,x_2 - 0.5\,x_3 \\
w_2 &= 1 + 2\,w_1 + 5\,x_2 \\
w_3 &= 0.5 + 1.5\,w_1 + 0.5\,x_2 - 0.5\,x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_3 = 1, \quad w_1 = x_2 = 0 \qquad\qquad x_1 = 2,\ w_2 = 1,\ w_3 = 0$$

For x_1, x_3 can become as large as $2.5/0.5 = 5$.

For w_2, x_3 can become as large as ∞.

For w_3, x_3 can become as large as $0.5/0.5 = 1$.

Step 3: Improving the solution

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 && -0.5\,w_1 && -1.5\,x_2 && -0.5\,x_3 \\
w_2 &= 1 && +2\,w_1 && +5\,x_2 && \\
w_3 &= 0.5 && +1.5\,w_1 && +0.5\,x_2 && -0.5\,\boxed{x_3}
\end{aligned}
$$

entering variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_3 = 1, \quad w_1 = x_2 = 0 \qquad\qquad x_1 = 2,\ w_2 = 1,\ w_3 = 0$$

For x_1, x_3 can become as large as $2.5/0.5 = 5$.

For w_2, x_3 can become as large as ∞.

For w_3, x_3 can become as large as $0.5/0.5 = 1$.

Step 3: Improving the solution

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

entering variable

$$
\begin{array}{rcl}
x_1 &=& 2.5 - 0.5\,w_1 - 1.5\,x_2 - 0.5\,x_3 \\
w_2 &=& 1 + 2\,w_1 + 5\,x_2 \\
\boxed{w_3} &=& 0.5 + 1.5\,w_1 + 0.5\,x_2 - 0.5\,\boxed{x_3}
\end{array}
$$

leaving variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_3 = 1, \quad w_1 = x_2 = 0 \qquad\qquad x_1 = 2,\ w_2 = 1,\ w_3 = 0$$

For x_1, x_3 can become as large as $2.5/0.5 = 5$.

For w_2, x_3 can become as large as ∞.

For w_3, x_3 can become as large as $0.5/0.5 = 1$.

The New Dictionary

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

The New Dictionary

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

basic variables

The New Dictionary

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$\begin{array}{rcrrr} x_1 & = & 2 & -2w_1 & -2x_2 & +w_3 \\ w_2 & = & 1 & +2w_1 & +5x_2 & \\ x_3 & = & 1 & +3w_1 & +x_2 & -2w_3 \end{array}$$

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

nonbasic variables

basic variables

The New Dictionary

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$\begin{aligned} x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3 \end{aligned}$$

nonbasic variables

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

basic variables

$$w_1 = 0,\ x_2 = 0\ w_3 = 0$$

The New Dictionary

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$\begin{array}{rcl} x_1 & = & 2 - 2w_1 - 2x_2 + w_3 \\ w_2 & = & 1 + 2w_1 + 5x_2 \\ x_3 & = & 1 + 3w_1 + x_2 - 2w_3 \end{array}$$

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

nonbasic variables

basic variables

$$w_1 = 0,\ x_2 = 0\ w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ w_3 = 1$$

The New Dictionary

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

the objective function value has increased

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

nonbasic variables

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

basic variables

$$w_1 = 0,\ x_2 = 0\ w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ w_3 = 1$$

The New Dictionary

Maximise $\underline{\zeta = 13} - w_1 - 2x_2 - w_3$

the objective function value has increased

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

nonbasic variables

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

basic variables

$$w_1 = 0,\ x_2 = 0\ w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ w_3 = 1$$

Which variable should we try to increase next?

The New Dictionary

Maximise $\zeta = \; 13 \; - \; w_1 \; -2 \; x_2 \; - \; w_3$

the objective function value has increased

subject to

$$\begin{array}{lcrlrlrl}
x_1 & = & 2 & -2\,w_1 & -2 & x_2 & + & w_3 \\
w_2 & = & 1 & +2\,w_1 & +5 & x_2 & & \\
x_3 & = & 1 & +3\,w_1 & + & x_2 & -2 & w_3
\end{array}$$

nonbasic variables

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

basic variables

$$w_1 = 0,\; x_2 = 0 \;\; w_3 = 0 \qquad x_1 = 2,\; w_2 = 1,\; w_3 = 1$$

Which variable should we try to increase next?

We have computed an optimal solution!

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

   3. Find a basic feasible solution by setting the nonbasic variables to 0.

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

 3. Find a basic feasible solution by setting the nonbasic variables to 0.

 4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i/\hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik}x_k \geq 0$).

5. Increase the value of the entering variable to be $x_k = \min_{i:\hat{a}_{ik}>0} \hat{b}_i/\hat{a}_{ik}$

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

5. Increase the value of the entering variable to be $x_k = \min_{i : \hat{a}_{ik} > 0} \hat{b}_i / \hat{a}_{ik}$

6. Compute the new dictionary making sure x_k only appears on the left.

Let’s do it again, “mechanically”

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$2x_1 + 3x_2 + x_3 \leq 5$$
$$4x_1 + x_2 + 2x + 3 \leq 11$$
$$3x_1 + 4x_2 + 2x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

Maximise $\zeta = 5x_1 + 4x_2 + 3x_3$

subject to

$$\begin{aligned} w_1 &= 5 - 2x_1 + 3x_2 + x_3 \\ w_2 &= 11 - 4x_1 + x_2 + 2x + 3 \\ w_3 &= 8 - 3x_1 + 4x_2 + 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

2. Write the original dictionary.

Maximise $\zeta = \quad +5\,x_1 \quad +4\,x_2 \quad +3\,x_3$

subject to

$$\begin{array}{rcrrrr}
w_1 & = & 5 & -2\,x_1 & -3\,x_2 & -\ \ x_3 \\
w_2 & = & 11 & -4\,x_1 & -\ \ x_2 & -2\,x_3 \\
w_3 & = & 8 & -3\,x_1 & -4\,x_2 & -2\,x_3
\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = \quad +5\ x_1 \quad +4\ x_2 \quad +3\ x_3$

subject to

$$\begin{aligned} w_1 &= 5 && -2\ x_1 && -3\ x_2 && -\ x_3 \\ w_2 &= 11 && -4\ x_1 && -\ x_2 && -2\ x_3 \\ w_3 &= 8 && -3\ x_1 && -4\ x_2 && -2\ x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = +5x_1 + 4x_2 + 3x_3$

subject to

$$
\begin{aligned}
w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\
w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\
w_3 &= 8 - 3x_1 - 4x_2 - 2x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = x_2 = x_3 = 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = +5x_1 + 4x_2 + 3x_3$

subject to

$$\begin{aligned} w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$$

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Maximise $\zeta = \quad +5x_1 \quad +4x_2 \quad +3x_3$

subject to

$$\begin{array}{rcrrrr}
w_1 & = & 5 & -2x_1 & -3x_2 & -x_3 \\
w_2 & = & 11 & -4x_1 & -x_2 & -2x_3 \\
w_3 & = & 8 & -3x_1 & -4x_2 & -2x_3
\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$x_1 = x_2 = x_3 = 0$ $\qquad$ $w_1 = 5, w_2 = 11, w_3 = 8$

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

$$
\begin{array}{llrrrr}
\textbf{Maximise} & \zeta = & & +5\ x_1 & +4\ x_2 & +3\ x_3 \\
\textbf{subject to} & w_1 = & 5 & -2\ x_1 & -3\ x_2 & -\ \ x_3 \\
 & w_2 = & 11 & -4\ x_1 & -\ \ x_2 & -2\ x_3 \\
 & w_3 = & 8 & -3\ x_1 & -4\ x_2 & -2\ x_3
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

$$
\begin{array}{llrrrr}
\textbf{Maximise} & \zeta = & & +5\ x_1 & +4\ x_2 & +3\ x_3 \\
\textbf{subject to} & w_1 = & 5 & -2\ \boxed{x_1} & -3\ x_2 & -\ x_3 \\
 & w_2 = & 11 & -4\ x_1 & -\ x_2 & -2\ x_3 \\
 & w_3 = & 8 & -3\ x_1 & -4\ x_2 & -2\ x_3 \\
 & & & \multicolumn{3}{l}{x_1, x_2, x_3, w_1, w_2, w_3 \geq 0}
\end{array}
$$

entering variable

$$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

$$\begin{array}{llrrrr} \textbf{Maximise} & \zeta = & & +5\ x_1 & +4\ x_2 & +3\ x_3 \\ & & & & & \\ \textbf{subject to} & w_1 = & 5 & -2\ \boxed{x_1} & -3\ x_2 & -\ x_3 \\ & w_2 = & 11 & -4\ x_1 & -\ x_2 & -2\ x_3 \\ & w_3 = & 8 & -3\ x_1 & -4\ x_2 & -2\ x_3 \end{array}$$

entering variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

entering variable

$$\begin{array}{lllrrr}
\textbf{Maximise} & \zeta = & & +5\ x_1 & +4\ x_2 & +3\ x_3 \\
\textbf{subject to} & w_1 = & 5 & -2\ \boxed{x_1} & -3\ x_2 & -\ x_3 \\
 & w_2 = & 11 & -4\ x_1 & -\ x_2 & -2\ x_3 \\
 & w_3 = & 8 & -3\ x_1 & -4\ x_2 & -2\ x_3
\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

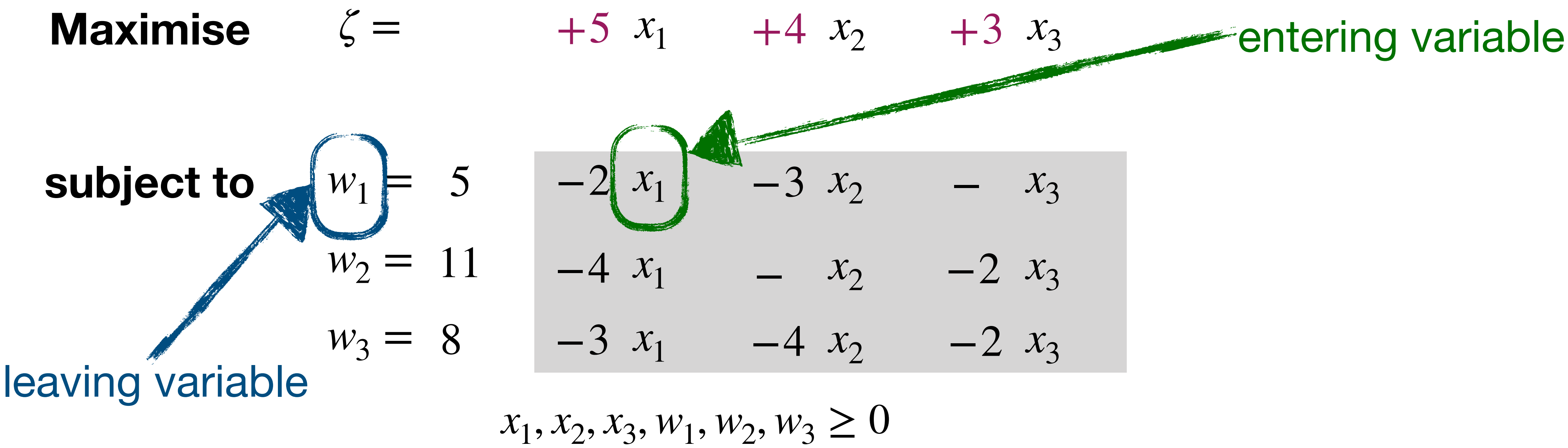
$$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

$5/2$ **vs** $11/4$ **vs** $8/3 \Rightarrow w_1$

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).



Maximise $\zeta = \quad +5\ x_1 \quad +4\ x_2 \quad +3\ x_3$

entering variable

subject to

$$
\begin{array}{rcrrrr}
\boxed{w_1} & = & 5 & -2\ \boxed{x_1} & -3\ x_2 & -\ x_3 \\
w_2 & = & 11 & -4\ x_1 & -\ x_2 & -2\ x_3 \\
w_3 & = & 8 & -3\ x_1 & -4\ x_2 & -2\ x_3
\end{array}
$$

leaving variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

$5/2$ **vs** $11/4$ **vs** $8/3 \Rightarrow w_1$

5. Increase the value of the entering variable to be $x_k = \min_{i:\hat{a}_{ik}>0} \hat{b}_i/\hat{a}_{ik}$

Maximise $\zeta = \quad +5\ x_1 \quad +4\ x_2 \quad +3\ x_3$

subject to

$$\begin{aligned} w_1 &= 5 && -2\ x_1 && -3\ x_2 && -\ \ x_3 \\ w_2 &= 11 && -4\ x_1 && -\ \ x_2 && -2\ x_3 \\ w_3 &= 8 && -3\ x_1 && -4\ x_2 && -2\ x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = 2.5,\ x_2 = 0,\ x_3 = 0$$

6. Compute the new dictionary making sure x_k only appears on the left.

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$\begin{aligned}
x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\
w_2 &= 1 + 2\, w_1 + 5\, x_2 \\
w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3
\end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

5. Increase the value of the entering variable to be $x_k = \min_{i : \hat{a}_{ik} > 0} \hat{b}_i / \hat{a}_{ik}$

6. Compute the new dictionary making sure x_k only appears on the left.

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

5. Increase the value of the entering variable to be $x_k = \min_{i : \hat{a}_{ik} > 0} \hat{b}_i / \hat{a}_{ik}$

6. Compute the new dictionary making sure x_k only appears on the left.

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

   3. Find a basic feasible solution by setting the nonbasic variables to 0.

   4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

   Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

   Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

   5. Increase the value of the entering variable to be $x_k = \min_{i:\hat{a}_{ik}>0} \hat{b}_i / \hat{a}_{ik}$

   6. Compute the new dictionary making sure x_k only appears on the left.

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5 w_1 - 1.5\, x_2 - 0.5\, x_3 \\ w_2 &= 1 + 2\, w_1 + 5\, x_2 \\ w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\ w_2 &= 1 + 2\, w_1 + 5\, x_2 \\ w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$\begin{array}{rcl|rrr} x_1 & = & 2.5 & -0.5\, w_1 & -1.5\, x_2 & -0.5\, x_3 \\ w_2 & = & 1 & +2\, w_1 & +5\, x_2 & \\ w_3 & = & 0.5 & +1.5\, w_1 & +0.5\, x_2 & -0.5\, x_3 \end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\
w_2 &= 1 + 2\, w_1 + 5\, x_2 \\
w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\ w_2 &= 1 + 2\, w_1 + 5\, x_2 \\ w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\ w_2 &= 1 + 2\, w_1 + 5\, x_2 \\ w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, \boxed{x_3} \end{aligned}$$

entering variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5\,w_1 - 1.5\,x_2 - 0.5\,x_3 \\ w_2 &= 1 + 2\,w_1 + 5\,x_2 \\ w_3 &= 0.5 + 1.5\,w_1 + 0.5\,x_2 - 0.5\,\boxed{x_3} \end{aligned}$$

entering variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Maximise $\zeta = \quad 12.5 \; -2.5\, w_1 \quad -3.5\, x_2 \quad +0.5\, x_3$

subject to

$$\begin{array}{rcllll} x_1 & = & 2.5 & -0.5\, w_1 & -1.5\, x_2 & -0.5\, x_3 \\ w_2 & = & 1 & +2\; w_1 & +5\; x_2 & \\ w_3 & = & 0.5 & +1.5\, w_1 & +0.5\, x_2 & -0.5\, \boxed{x_3} \end{array}$$

entering variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0 \qquad x_1 = 2.5,\; w_2 = 1,\; w_3 = 0.5$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

$2.5/0.5$ **vs** ∞ **vs** $0.5/0.5 \Rightarrow w_3$

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$\begin{array}{rrcrrrrl}
 & x_1 & = & 2.5 & -0.5\, w_1 & -1.5\, x_2 & -0.5\, x_3 & \\
 & w_2 & = & 1 & +2\, w_1 & +5\, x_2 & & \\
\text{leaving variable} \rightarrow & \boxed{w_3} & = & 0.5 & +1.5\, w_1 & +0.5\, x_2 & -0.5\, \boxed{x_3} & \leftarrow \text{entering variable}
\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

$2.5/0.5$ **vs** ∞ **vs** $0.5/0.5 \Rightarrow w_3$

5. Increase the value of the entering variable to be $x_k = \min\limits_{i:\hat{a}_{ik}>0} \hat{b}_i/\hat{a}_{ik}$

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$\begin{array}{rcrrrr} x_1 & = & 2.5 & -0.5\,w_1 & -1.5\,x_2 & -0.5\,x_3 \\ w_2 & = & 1 & +2\,w_1 & +5\,x_2 & \\ w_3 & = & 0.5 & +1.5\,w_1 & +0.5\,x_2 & -0.5\,x_3 \end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = 2.5,\ x_2 = 0,\ x_3 = 1$$

6. Compute the new dictionary making sure x_k only appears on the left.

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$\begin{aligned} x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = w_3 = 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ w_3 = 1$$

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

  3. Find a basic feasible solution by setting the nonbasic variables to 0.

  4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

  Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

  Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

  5. Increase the value of the entering variable to be $x_k = \min_{i : \hat{a}_{ik} > 0} \hat{b}_i / \hat{a}_{ik}$

  6. Compute the new dictionary making sure x_k only appears on the left.

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

5. Increase the value of the entering variable to be $x_k = \min_{i : \hat{a}_{ik} > 0} \hat{b}_i / \hat{a}_{ik}$

6. Compute the new dictionary making sure x_k only appears on the left.

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

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5. Increase the value of the entering variable to be $x_k = \min_{i : \hat{a}_{ik} > 0} \hat{b}_i / \hat{a}_{ik}$

6. Compute the new dictionary making sure x_k only appears on the left.

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ w_3 = 1$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ w_3 = 1$$

We have computed an optimal solution!

Potential Problem

Potential Problem

Consider the following LP:

Potential Problem

Consider the following LP:

Maximise $-2x_1 - x_2$

subject to

$$\begin{aligned} -x_1 + x_2 &\leq -1 \\ -x_1 - 2x_2 &\leq -2 \\ x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Corresponding dictionary

Maximise $\zeta = \quad -2\,x_1 \quad - x_2$

subject to

$$\begin{aligned} w_1 &= -1 + x_1 - x_2 \\ w_2 &= -2 + x_1 + 2\,x_2 \\ w_3 &= 1 - x_2 \end{aligned}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

Corresponding dictionary

Maximise $\zeta = \quad -2\ x_1 \quad - x_2$

subject to

$$
\begin{aligned}
w_1 &= -1 + x_1 - x_2 \\
w_2 &= -2 + x_1 + 2\ x_2 \\
w_3 &= 1 - x_2
\end{aligned}
$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Corresponding dictionary

Maximise $\zeta = -2x_1 - x_2$

subject to

$$\begin{array}{rcrrr} w_1 & = & -1 & + x_1 & - x_2 \\ w_2 & = & -2 & + x_1 & +2x_2 \\ w_3 & = & 1 & & - x_2 \end{array}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = x_2 = x_3 = 0$$

Corresponding dictionary

Maximise $\zeta = \quad -2\,x_1 \quad -x_2$

subject to

$$\begin{array}{rcrcrcr} w_1 & = & -1 & + & x_1 & - & x_2 \\ w_2 & = & -2 & + & x_1 & +2 & x_2 \\ w_3 & = & 1 & & & - & x_2 \end{array}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = x_2 = x_3 = 0 \qquad w_1 = -1,\ w_2 = -2,\ w_3 = 1$$

Corresponding dictionary

Maximise $\zeta = -2\ x_1 - x_2$

subject to

$$\begin{aligned} w_1 &= -1 + x_1 - x_2 \\ w_2 &= -2 + x_1 + 2\ x_2 \\ w_3 &= 1 - x_2 \end{aligned}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = x_2 = x_3 = 0 \qquad\qquad w_1 = -1,\ w_2 = -2,\ w_3 = 1$$

The dictionary is infeasible!

Corresponding dictionary

Maximise $\zeta = \quad -2\ x_1 \quad -x_2$

subject to

$$\begin{array}{rcrrr} w_1 &=& -1 & +\ x_1 & -\ x_2 \\ w_2 &=& -2 & +\ x_1 & +2\ x_2 \\ w_3 &=& 1 & & -\ x_2 \end{array}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = x_2 = x_3 = 0 \qquad\qquad w_1 = -1,\ w_2 = -2,\ w_3 = 1$$

The dictionary is infeasible!

This can be handled via an appropriate “trick”, see Vanderbei Chapter 2.

Initialisation

Consider the following LP:

Maximise $-2x_1 - x_2$

subject to

$$
\begin{aligned}
-x_1 + x_2 &\leq -1 \\
-x_1 - 2x_2 &\leq -2 \\
x_2 &\leq 1 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Initialisation

Consider the following alternative LP:

Maximise $-x_0$

subject to

$$-x_1 + x_2 - x_0 \leq -1$$

$$-x_1 - 2x_2 - x_0 \leq -2$$

$$x_2 - x_0 \leq 1$$

$$x_1, x_2, x_0 \geq 0$$

Initialisation

subject to

$$
\begin{aligned}
-x_1 + x_2 &\leq -1 \\
-x_1 - 2x_2 &\leq -2 \\
x_2 &\leq 1 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Maximise $-x_0$

subject to

$$
\begin{aligned}
-x_1 + x_2 - x_0 &\leq -1 \\
-x_1 - 2x_2 - x_0 &\leq -2 \\
x_2 - x_0 &\leq 1 \\
x_1, x_2, x_0 &\geq 0
\end{aligned}
$$

Initialisation

subject to
$$-x_1 + x_2 \leq -1$$
$$-x_1 - 2x_2 \leq -2$$
$$x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

The first LP is feasible if any only if the second LP has an optimal solution of value 0.

Maximise $-x_0$

subject to
$$-x_1 + x_2 - x_0 \leq -1$$
$$-x_1 - 2x_2 - x_0 \leq -2$$
$$x_2 - x_0 \leq 1$$
$$x_1, x_2, x_0 \geq 0$$

Initialisation

Consider the following alternative LP:

Maximise $- x_0$

subject to

$$
\begin{aligned}
-x_1 + x_2 - x_0 &\leq -1 \\
-x_1 - 2x_2 - x_0 &\leq -2 \\
x_2 - x_0 &\leq 1 \\
x_1, x_2, x_0 &\geq 0
\end{aligned}
$$

Auxiliary problem dictionary

Maximise $\zeta = \qquad - x_0$

subject to

$$\begin{aligned} w_1 &= -1 + x_1 - x_2 + x_0 \\ w_2 &= -2 + x_1 + 2x_2 + x_0 \\ w_3 &= 1 - x_2 + x_0 \end{aligned}$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

Auxiliary problem dictionary

Maximise $\zeta = -x_0$

subject to

$$\begin{aligned} w_1 &= -1 + x_1 - x_2 + x_0 \\ w_2 &= -2 + x_1 + 2x_2 + x_0 \\ w_3 &= 1 - x_2 + x_0 \end{aligned}$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Auxiliary problem dictionary

Maximise $\zeta = \quad - x_0$

subject to

$$\begin{aligned} w_1 &= -1 + x_1 - x_2 + x_0 \\ w_2 &= -2 + x_1 + 2x_2 + x_0 \\ w_3 &= 1 - x_2 + x_0 \end{aligned}$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

The dictionary is infeasible!

Auxiliary problem dictionary

Maximise $\zeta = \qquad - x_0$

subject to

$$
\begin{aligned}
w_1 &= -1 + x_1 - x_2 + x_0 \\
w_2 &= -2 + x_1 + 2x_2 + x_0 \\
w_3 &= 1 - x_2 + x_0
\end{aligned}
$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

The dictionary is infeasible!

Entering variable: x_0

Auxiliary problem dictionary

Maximise $\zeta = \qquad - x_0$

subject to

$$\begin{aligned} w_1 &= -1 + x_1 - x_2 + x_0 \\ w_2 &= -2 + x_1 + 2x_2 + \boxed{x_0} \\ w_3 &= 1 - x_2 + x_0 \end{aligned}$$

entering variable

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

The dictionary is infeasible!

Entering variable: x_0

Auxiliary problem dictionary

Maximise $\zeta = -x_0$

subject to

$$\begin{aligned} w_1 &= -1 + x_1 - x_2 + x_0 \\ w_2 &= -2 + x_1 + 2x_2 + \boxed{x_0} \\ w_3 &= 1 - x_2 + x_0 \end{aligned}$$

entering variable

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

The dictionary is infeasible!

Entering variable: x_0

Leaving variable: the one that is “most infeasible”

Auxiliary problem dictionary

$$\begin{array}{llrrrr}
\textbf{Maximise} & \zeta = & & & & -x_0 \\
\textbf{subject to} & w_1 = & -1 & +\ x_1 & -\ x_2 & +\ x_0 \\
 & \boxed{w_2} = & -2 & +\ x_1 & +2\ x_2 & +\ \boxed{x_0} \\
 & w_3 = & 1 & & -\ x_2 & +\ x_0
\end{array}$$

leaving variable

entering variable

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

The dictionary is infeasible!

Entering variable: x_0

Leaving variable: the one that is “most infeasible”

Auxiliary problem dictionary

Maximise $\zeta = -x_0$

subject to

entering variable

leaving variable

$$\begin{array}{rcrrrr} w_1 &=& -1 & + \; x_1 & - \; x_2 & + \; x_0 \\ \boxed{w_2} &=& -2 & + \; x_1 & +2 \; x_2 & + \; \boxed{x_0} \\ w_3 &=& 1 & & - \; x_2 & + \; x_0 \end{array}$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

The dictionary is infeasible!

Entering variable: x_0

Leaving variable: the one that is “most infeasible”

6. Compute the new dictionary making sure x_0 only appears on the left.

The new auxiliary problem dictionary

Maximise $\zeta = -2 + x_1 + 2x_2 - w_2$

subject to

$$
\begin{aligned}
w_1 &= 1 \phantom{{}- x_1} - 3x_2 + w_2 \\
x_0 &= 2 - x_1 - 2x_2 + w_2 \\
w_3 &= 3 - x_1 - 3x_2 + w_2
\end{aligned}
$$

$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$

The new auxiliary problem dictionary

Maximise $\zeta = -2 + x_1 + 2x_2 - w_2$

subject to

$$
\begin{aligned}
w_1 &= 1 \qquad\qquad\;\, -3x_2 + w_2\\
x_0 &= 2 - x_1 - 2x_2 + w_2\\
w_3 &= 3 - x_1 - 3x_2 + w_2
\end{aligned}
$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

The dictionary is feasible, we can apply the simplex method.

The new auxiliary problem dictionary

Maximise $\zeta = -2 + x_1 + 2x_2 - w_2$

subject to

$$
\begin{array}{rcrrrr}
w_1 & = & 1 & & -3x_2 & + w_2 \\
x_0 & = & 2 & - x_1 & -2x_2 & + w_2 \\
w_3 & = & 3 & - x_1 & -3x_2 & + w_2
\end{array}
$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

The dictionary is feasible, we can apply the simplex method.

steps…

The final auxiliary problem dictionary

Maximise $\zeta = -x_0$

subject to

$$x_2 = 0.33 \qquad\qquad -0.33\,w_1 + 0.33\,w_2$$

$$x_1 = 1.33 - x_0 + 0.67\,w_1 + 0.33\,w_2$$

$$w_3 = 2 + x_0 + 0.33\,w_1 + 0.33\,w_2$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

The final auxiliary problem dictionary

Maximise $\zeta = -x_0$

subject to

$$\begin{aligned} x_2 &= 0.33 & & & -0.33\,w_1 & +0.33\,w_2 \\ x_1 &= 1.33 & - & x_0 & +0.67\,w_1 & +0.33\,w_2 \\ w_3 &= 2 & + & x_0 & +0.33\,w_1 & +0.33\,w_2 \end{aligned}$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

Remove x_0 from the constraints and substitute the original objective function.

The first dictionary of our original problem

Maximise $\zeta = -2\,x_1 - x_2$

subject to

$$
\begin{aligned}
x_2 &= 0.33 - 0.33\,w_1 + 0.33\,w_2 \\
x_1 &= 1.33 + 0.67\,w_1 + 0.33\,w_2 \\
w_3 &= 2 + 0.33\,w_1 + 0.33\,w_2
\end{aligned}
$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

The first dictionary of our original problem

Maximise $\zeta = -2\,x_1 - x_2$

subject to

$$x_2 = 0.33 - 0.33\,w_1 + 0.33\,w_2$$

$$x_1 = 1.33 + 0.67\,w_1 + 0.33\,w_2$$

$$w_3 = 2 + 0.33\,w_1 + 0.33\,w_2$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

We should have only nonbasic variables in the objective function.

Easy Fix

Maximise $\zeta = -2x_1 - x_2$

subject to

$$
\begin{aligned}
w_1 &= -1 + x_1 - x_2 \\
w_2 &= -2 + x_1 + 2x_2 \\
w_3 &= 1 - x_2
\end{aligned}
$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

Easy Fix

Maximise $\zeta = -2x_1 - x_2$

subject to

$$\begin{array}{rcrrr} w_1 & = & -1 & + x_1 & - x_2 \\ w_2 & = & -2 & + x_1 & +2x_2 \\ w_3 & = & 1 & & - x_2 \end{array}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

We have $\zeta = -2x_1 - x_2 = -3 - w_1 - w_2$

The first dictionary of our original problem

Maximise $\zeta = \quad -3 \; w_1 \; - \; w_2$

subject to

$x_2 = 0.33 \; -0.33\, w_1 \; +0.33\, w_2$

$x_1 = 1.33 \; +0.67\, w_1 \; +0.33\, w_2$

$w_3 = 2 \; +0.33\, w_1 \; +0.33\, w_2$

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

The first dictionary of our original problem

Maximise $\zeta = \quad -3\ w_1 \quad - \quad w_2$

subject to

$$
\begin{aligned}
x_2 &= 0.33 \quad -0.33\,w_1 \quad +0.33\,w_2 \\
x_1 &= 1.33 \quad +0.67\,w_1 \quad +0.33\,w_2 \\
w_3 &= 2 \quad +0.33\,w_1 \quad +0.33\,w_2
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

The first dictionary of our original problem

Maximise $\zeta = -3\,w_1 - w_2$

subject to

$$\begin{aligned} x_2 &= 0.33 - 0.33\,w_1 + 0.33\,w_2 \\ x_1 &= 1.33 + 0.67\,w_1 + 0.33\,w_2 \\ w_3 &= 2 + 0.33\,w_1 + 0.33\,w_2 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = w_2 = 0$$

The first dictionary of our original problem

Maximise $\zeta = -3\,w_1 - w_2$

subject to

$$\begin{aligned} x_2 &= 0.33 - 0.33\,w_1 + 0.33\,w_2 \\ x_1 &= 1.33 + 0.67\,w_1 + 0.33\,w_2 \\ w_3 &= 2 + 0.33\,w_1 + 0.33\,w_2 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = w_2 = 0 \qquad x_1 = 1.33,\ x_2 = 0.33,\ w_3 = 2$$

The first dictionary of our original problem

Maximise $\zeta = -3\,w_1 - w_2$

subject to

$$\begin{array}{rcrrr} x_2 & = & 0.33 & -0.33\,w_1 & +0.33\,w_2 \\ x_1 & = & 1.33 & +0.67\,w_1 & +0.33\,w_2 \\ w_3 & = & 2 & +0.33\,w_1 & +0.33\,w_2 \end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = w_2 = 0 \qquad x_1 = 1.33,\ x_2 = 0.33,\ w_3 = 2$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

The first dictionary of our original problem

Maximise $\zeta = -3\ w_1 - w_2$

subject to

$$\begin{array}{rl} x_2 = & 0.33 \quad -0.33\,w_1 \ +0.33\,w_2 \\ x_1 = & 1.33 \quad +0.67\,w_1 \ +0.33\,w_2 \\ w_3 = & 2 \quad +0.33\,w_1 \ +0.33\,w_2 \end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

We have found an optimal solution!

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = w_2 = 0 \qquad x_1 = 1.33,\ x_2 = 0.33,\ w_3 = 2$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

The first dictionary of our original problem

Maximise $\zeta = \qquad -3\ w_1 \quad - \quad w_2$

subject to

$$\begin{aligned} x_2 &= 0.33 & -0.33\,w_1 & +0.33\,w_2 \\ x_1 &= 1.33 & +0.67\,w_1 & +0.33\,w_2 \\ w_3 &= 2 & +0.33\,w_1 & +0.33\,w_2 \end{aligned}$$

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

We have found an optimal solution!

We were lucky: we can only expect to find a feasible solution.

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$w_1 = w_2 = 0 \qquad x_1 = 1.33,\ x_2 = 0.33,\ w_3 = 2$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

What if we have this dictionary?

Maximise $\zeta = 5 + x_3 - x_1$

subject to

$$
\begin{aligned}
x_2 &= 5 + 2x_3 - 3x_1 \\
x_4 &= 7 - 4x_1 \\
x_5 &= x_1
\end{aligned}
$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

What if we have this dictionary?

Maximise $\zeta = 5 + x_3 - x_1$

subject to

$$\begin{aligned} x_2 &= 5 + 2x_3 - 3x_1 \\ x_4 &= 7 - 4x_1 \\ x_5 &= x_1 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

What if we have this dictionary?

$$
\begin{array}{llcccccc}
\textbf{Maximise} & \zeta = & 5 & + & x_3 & - & x_1 \\
\\
\textbf{subject to} & x_2 = & 5 & +2 & x_3 & -3 & x_1 \\
 & x_4 = & 7 & & & -4 & x_1 \\
 & x_5 = & & & & & x_1 \\
\end{array}
$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

What if we have this dictionary?

entering variable

$$
\begin{array}{llrrr}
\textbf{Maximise} & \zeta = & 5 & +\boxed{x_3} & -x_1 \\
 & & & & \\
\textbf{subject to} & x_2 = & 5 & +2x_3 & -3x_1 \\
 & x_4 = & 7 & & -4x_1 \\
 & x_5 = & & & x_1
\end{array}
$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

What if we have this dictionary?

entering variable

Maximise $\zeta = 5 + \boxed{x_3} - x_1$

subject to

$$\begin{array}{rrrr} x_2 = & 5 & +2\ x_3 & -3\ x_1 \\ x_4 = & 7 & & -4\ x_1 \\ x_5 = & & & x_1 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

What if we have this dictionary?

entering variable

Maximise $\zeta = 5 + \boxed{x_3} - x_1$

subject to

$$\begin{aligned} x_2 &= 5 + 2x_3 - 3x_1 \\ x_4 &= 7 - 4x_1 \\ x_5 &= x_1 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

We can increase the value of some nonbasic variable, here x_3

What if we have this dictionary?

entering variable

$$\textbf{Maximise} \quad \zeta = 5 + \boxed{x_3} - x_1$$

$$\begin{aligned} \textbf{subject to} \quad x_2 &= 5 + 2x_3 - 3x_1 \\ x_4 &= 7 - 4x_1 \\ x_5 &= x_1 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

We can increase the value of some nonbasic variable, here x_3

We should not violate any constraints though!

What if we have this dictionary?

entering variable

Maximise $\zeta = 5 + \boxed{x_3} - x_1$

subject to

$$\begin{aligned} x_2 &= 5 + 2x_3 - 3x_1 \\ x_4 &= 7 - 4x_1 \\ x_5 &= x_1 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

We can increase the value of some nonbasic variable, here x_3

We should not violate any constraints though!

We don’t want any of the slack variables to become negative.

What if we have this dictionary?

entering variable

Maximise $$\zeta = 5 + \boxed{x_3} - x_1$$

subject to
$$\begin{aligned} x_2 &= 5 + 2\,x_3 - 3\,x_1 \\ x_4 &= 7 - 4\,x_1 \\ x_5 &= x_1 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

We can increase the value of some nonbasic variable, here x_3

We should not violate any constraints though!

We don't want any of the slack variables to become negative.

What if we have this dictionary?

entering variable

$$\begin{array}{llrrrrrr}
\textbf{Maximise} & \zeta = & 5 & + & & \boxed{x_3} & - & x_1 \\
\textbf{subject to} & x_2 = & 5 & +2 & & x_3 & -3 & x_1 \\
 & x_4 = & 7 & & & & -4 & x_1 \\
 & x_5 = & & & & & & x_1
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

We can increase the value of some nonbasic variable, here x_3

We should not violate any constraints though!

We don’t want any of the slack variables to become negative.

This does not happen regardless of how much we increase x_3.

What if we have this dictionary?

entering variable

$$
\begin{array}{llrrrrr}
\textbf{Maximise} & \zeta = & 5 & + & \boxed{x_3} & - & x_1 \\
\textbf{subject to} & x_2 = & 5 & +2 & x_3 & -3 & x_1 \\
 & x_4 = & 7 & & & -4 & x_1 \\
 & x_5 = & & & & & x_1 \\
 & & & & x_1, x_2, x_3, x_4, x_5 \geq 0 & &
\end{array}
$$

The LP is unbounded!

We can increase the value of some nonbasic variable, here x_3

We should not violate any constraints though!

We don’t want any of the slack variables to become negative.

This does not happen regardless of how much we increase x_3.

What about this dictionary?

Maximise $\zeta = 3 \quad -0.5\,x_1 \quad +2\,x_2 \quad -1.5\,w_1$

subject to

$$\begin{array}{rllll} x_3 = & 1 & -0.5\,x_1 & & -0.5\,w_1 \\ w_2 = & & x_1 & -\ x_2 & +\ w_1 \end{array}$$

$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

What about this dictionary?

$$\begin{array}{llrrrr}
\textbf{Maximise} & \zeta = & 3 & -0.5\,x_1 & +2\,x_2 & -1.5\,w_1 \\
\textbf{subject to} & x_3 = & 1 & -0.5\,x_1 & & -0.5\,w_1 \\
 & w_2 = & & x_1 & -\boxed{x_2} & +\,w_1
\end{array}$$

entering variable

$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

What about this dictionary?

Maximise $\zeta = 3 - 0.5\,x_1 + 2\,x_2 - 1.5\,w_1$

subject to

$$\begin{aligned} x_3 &= 1 - 0.5\,x_1 - 0.5\,w_1 \\ \boxed{w_2} &= x_1 - \boxed{x_2} + w_1 \end{aligned}$$

entering variable

leaving variable

$x_1, x_2, x_3, w_1, w_2 \geq 0$

What about this dictionary?

$$
\begin{array}{llcl}
\textbf{Maximise} & \zeta = & 3 & -0.5\,x_1 + 2\,x_2 - 1.5\,w_1 \\
\textbf{subject to} & x_3 = & 1 & -0.5\,x_1 \qquad\qquad -0.5\,w_1 \\
 & \boxed{w_2} = & & x_1 - \boxed{x_2} + w_1 \\
 & & & x_1, x_2, x_3, w_1, w_2 \geq 0
\end{array}
$$

entering variable

leaving variable

We can increase the value of some nonbasic variable, here x_2

We should not violate any constraints though!

We don’t want any of the slack variables to become negative.

What about this dictionary?

Maximise $\zeta = 3 \quad -0.5\,x_1 \quad +2 \quad x_2 \quad -1.5\,w_1$

subject to $x_3 = 1 \quad -0.5\,x_1 \quad -0.5\,w_1$ entering variable

leaving variable $\boxed{w_2} = \quad x_1 \quad - \quad \boxed{x_2} \quad + \quad w_1$

$x_1, x_2, x_3, w_1, w_2 \geq 0$

We can increase the value of some nonbasic variable, here x_2

We should not violate any constraints though!

We don't want any of the slack variables to become negative.

x_2 cannot be increased! Are we stuck?

Degeneracy

Degeneracy

Degenerate dictionary: A dictionary in which one of the b_i variables becomes zero.

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Degenerate dictionary: A dictionary in which one of the b_i variables becomes zero.

Equivalently: In a basic feasible solution, one of the basic variables is 0.

Degeneracy not necessarily and issue

entering variable

Maximise $$\zeta = 5 + \boxed{x_3} - x_1$$

subject to

$$x_2 = 5 + 2x_3 - 3x_1$$

$$x_4 = 7 - 4x_1$$

$$x_5 = x_1$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The LP is unbounded!

We can increase the value of some nonbasic variable, here x_3

We should not violate any constraints though!

$\underline{\text{We don’t want any of the slack variables to become negative.}}$

This does not happen regardless of how much we increase x_3.

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Degenerate Pivot: The entering variable stays at 0 without increasing.

What about this dictionary?

$$
\begin{array}{llllllll}
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\textbf{subject to} & x_3 = & 1 & -0.5\,x_1 & & & -0.5\,w_1 \\
 & \boxed{w_2} = & & x_1 & - & \boxed{x_2} & + & w_1
\end{array}
$$

entering variable

leaving variable

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Degenerate Pivot: The entering variable stays at 0 without increasing.

"Degenerate pivots are quite common and usually harmless."

Let’s not give up

Maximise $\zeta = 3 - 0.5\,x_1 + 2\,x_2 - 1.5\,w_1$

subject to

$$\begin{aligned} x_3 &= 1 - 0.5\,x_1 - 0.5\,w_1 \\ \boxed{w_2} &= x_1 - \boxed{x_2} + w_1 \end{aligned}$$

entering variable

leaving variable

$x_1, x_2, x_3, w_1, w_2 \geq 0$

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subject to $x_3 = 1 - 0.5\,x_1 - 0.5\,w_1$ entering variable

leaving variable $\boxed{w_2} = x_1 - \boxed{x_2} + w_1$

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We should not violate any constraints though!

We don’t want any of the slack variables to become negative.

Increase the variable as much as we can (as before): here 0 increase

Let’s not give up

Maximise $\zeta = 3 \quad -0.5\,x_1 \quad +2\,x_2 \quad -1.5\,w_1$

subject to $x_3 = 1 \quad -0.5\,x_1 \qquad -0.5\,w_1$

entering variable

leaving variable $\boxed{w_2} = \quad x_1 \quad - \boxed{x_2} \quad + \quad w_1$

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Actually pivot!

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$x_1, x_2, x_3, w_1, w_2 \geq 0$

Actually pivot!

$(x_1, x_2, x_3, w_1, w_2) = (0,0,1,0,0)$

We can increase the value of some nonbasic variable, here x_2

We should not violate any constraints though!

We don’t want any of the slack variables to become negative.

Increase the variable as much as we can (as before): here 0 increase

The new dictionary

Maximise $\zeta = 3 + 1.5\, x_1 + 2\, w_2 + 0.5\, w_1$

subject to

$$
\begin{aligned}
x_3 &= 1 - 0.5\, x_1 - 0.5\, w_1 \\
x_2 &= x_1 - w_2 + w_1
\end{aligned}
$$

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The new dictionary

Maximise $\zeta = 3 + 1.5\,x_1 + 2\,w_2 + 0.5\,w_1$

entering variable

subject to $\boxed{x_3} = 1 - 0.5\,\boxed{x_1} - 0.5\,w_1$

leaving variable $x_2 = x_1 - w_2 + w_1$

$x_1, x_2, x_3, w_1, w_2 \geq 0$

$(x_1, x_2, x_3, w_1, w_2) = (0,0,1,0,0)$

The new dictionary

Maximise $\zeta = 3 \quad +1.5\, x_1 \quad +2 \;\; w_2 \quad +0.5\, w_1$

entering variable

subject to $\boxed{x_3} = 1 \quad -0.5\, \boxed{x_1} \qquad\qquad -0.5\, w_1$

leaving variable

$x_2 = \qquad x_1 \quad - \quad w_2 \quad + \quad w_1$

$x_1, x_2, x_3, w_1, w_2 \geq 0$

$(x_1, x_2, x_3, w_1, w_2) = (0{,}0{,}1{,}0{,}0)$

We can now increase x_1 to $x_1 = 2$

The new dictionary

Maximise $\zeta = 3 + 1.5\,x_1 + 2\,w_2 + 0.5\,w_1$

entering variable

subject to $\boxed{x_3} = 1 - 0.5\,\boxed{x_1} - 0.5\,w_1$

leaving variable

$x_2 = x_1 - w_2 + w_1$

$x_1, x_2, x_3, w_1, w_2 \geq 0$

$(x_1, x_2, x_3, w_1, w_2) = (0{,}0{,}1{,}0{,}0)$

We can now increase x_1 to $x_1 = 2$

The pivot is not degenerate!

The new dictionary

Maximise $\zeta = 3 + 1.5\,x_1 + 2\,w_2 + 0.5\,w_1$

entering variable

subject to

$$\begin{aligned} \boxed{x_3} &= 1 - 0.5\,\boxed{x_1} - 0.5\,w_1 \\ x_2 &= x_1 - w_2 + w_1 \end{aligned}$$

leaving variable

$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

$$(x_1, x_2, x_3, w_1, w_2) = (0,0,1,0,0)$$

We can now increase x_1 to $x_1 = 2$

The pivot is not degenerate!

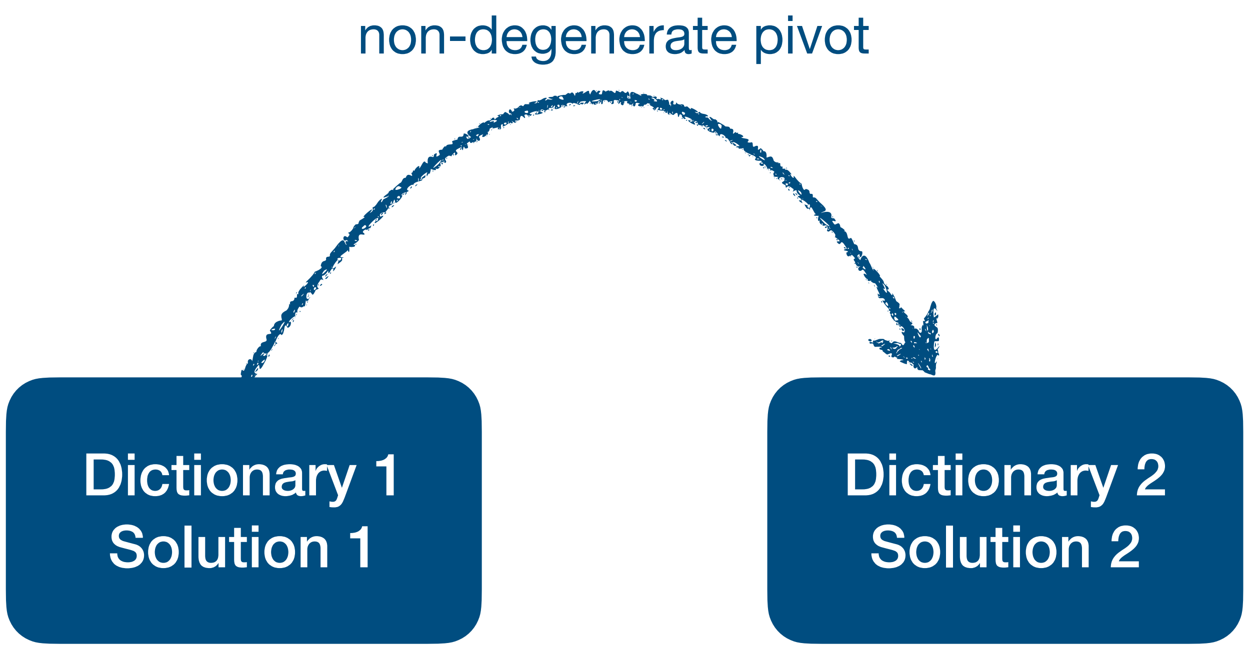
It will actually lead to a final dictionary, and an optimal solution.

Pictorially

Pictorially

Dictionary 1
Solution 1

Pictorially

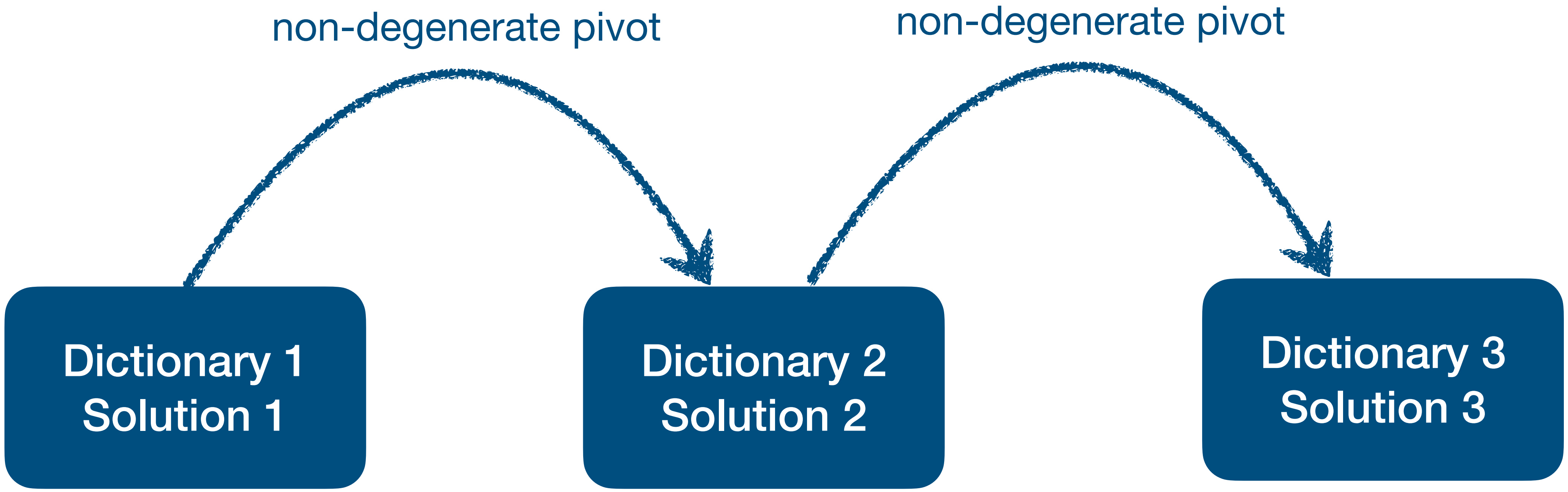


non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Pictorially



non-degenerate pivot

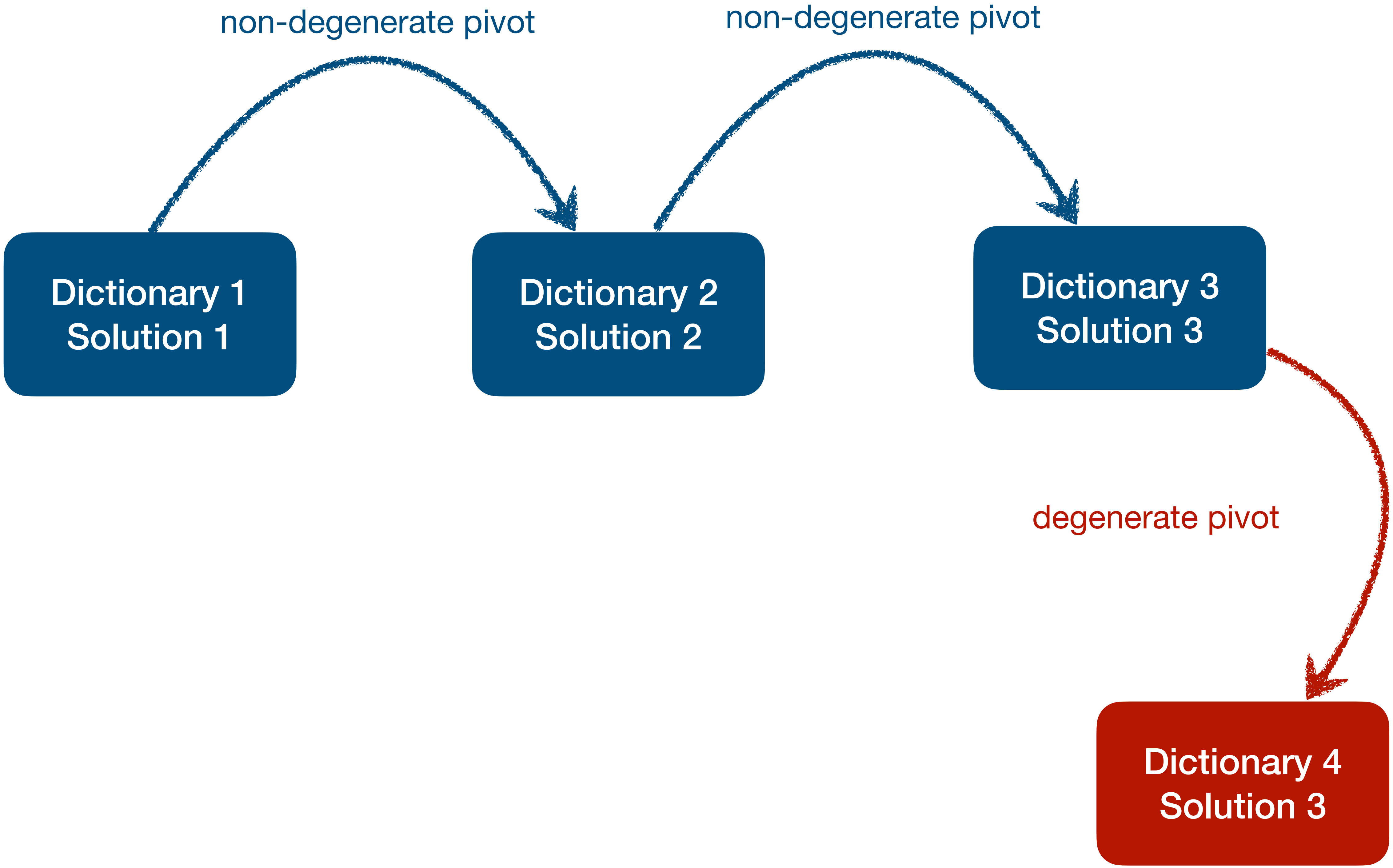
non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Dictionary 3
Solution 3

Pictorially


non-degenerate pivot

non-degenerate pivot

Dictionary 1
Solution 1

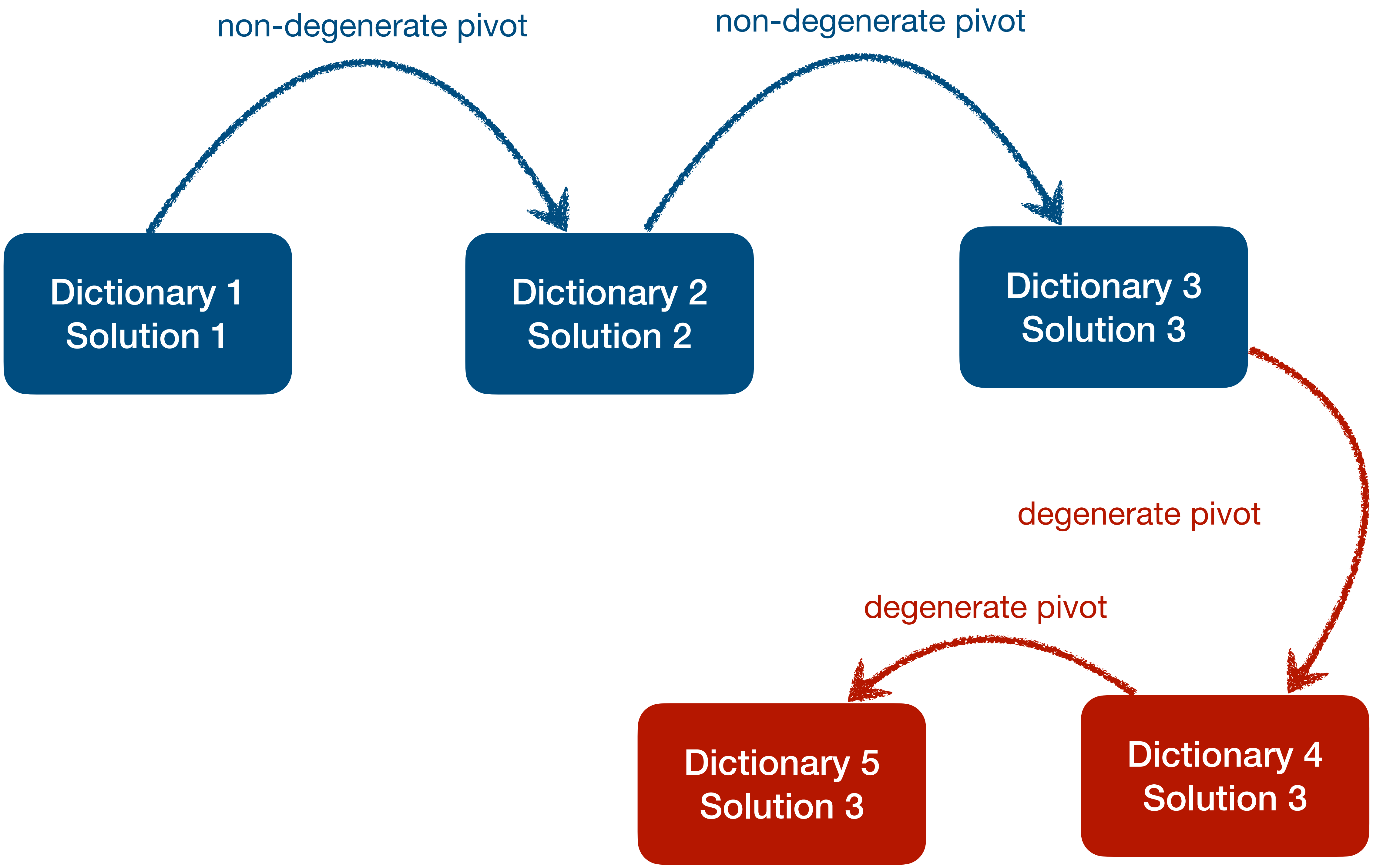
Dictionary 2
Solution 2

Dictionary 3
Solution 3

degenerate pivot

Dictionary 4
Solution 3

Pictorially



non-degenerate pivot

non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Dictionary 3
Solution 3

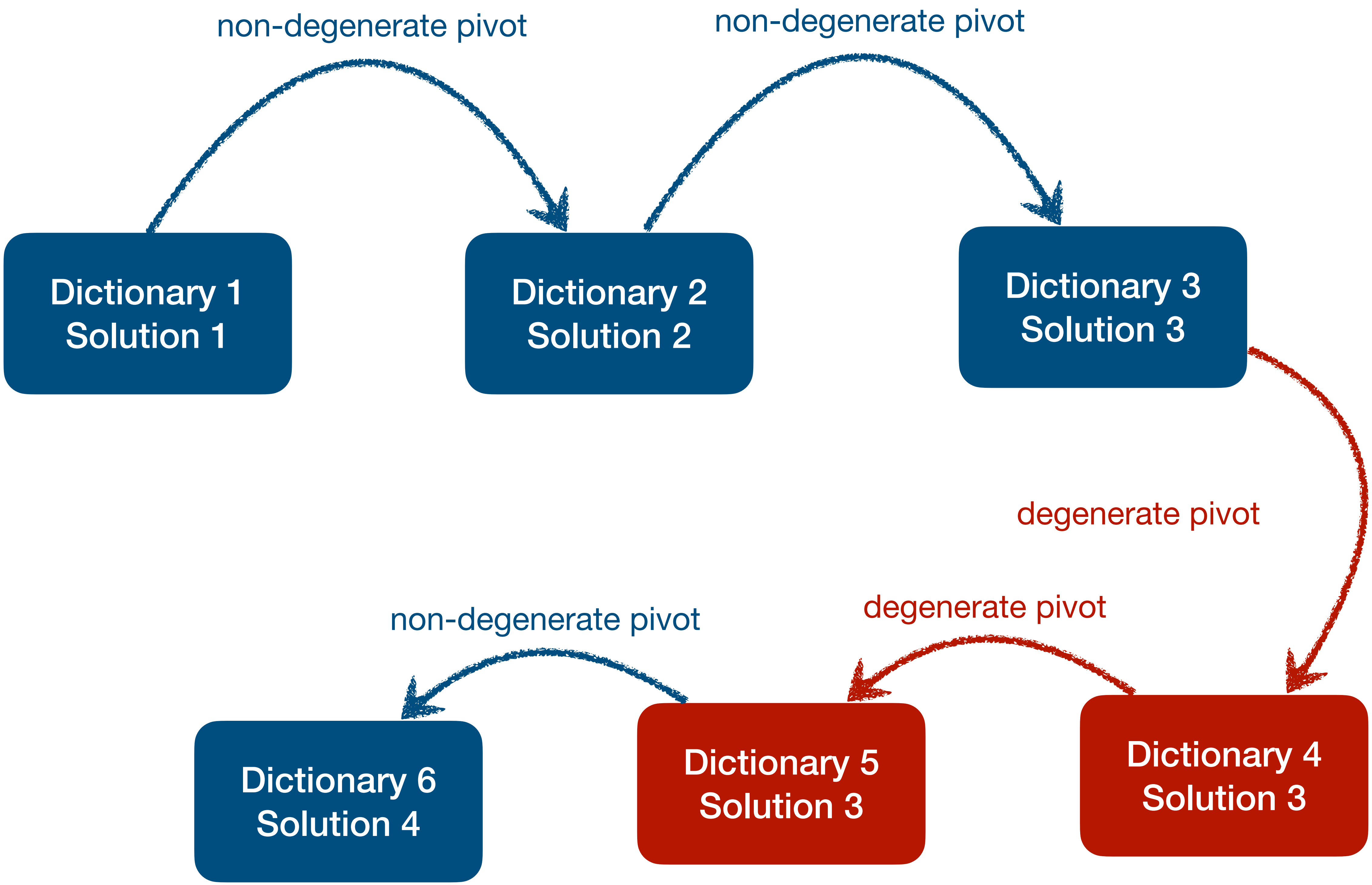
degenerate pivot

degenerate pivot

Dictionary 5
Solution 3

Dictionary 4
Solution 3

Pictorially



non-degenerate pivot

non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Dictionary 3
Solution 3

degenerate pivot

Dictionary 4
Solution 3

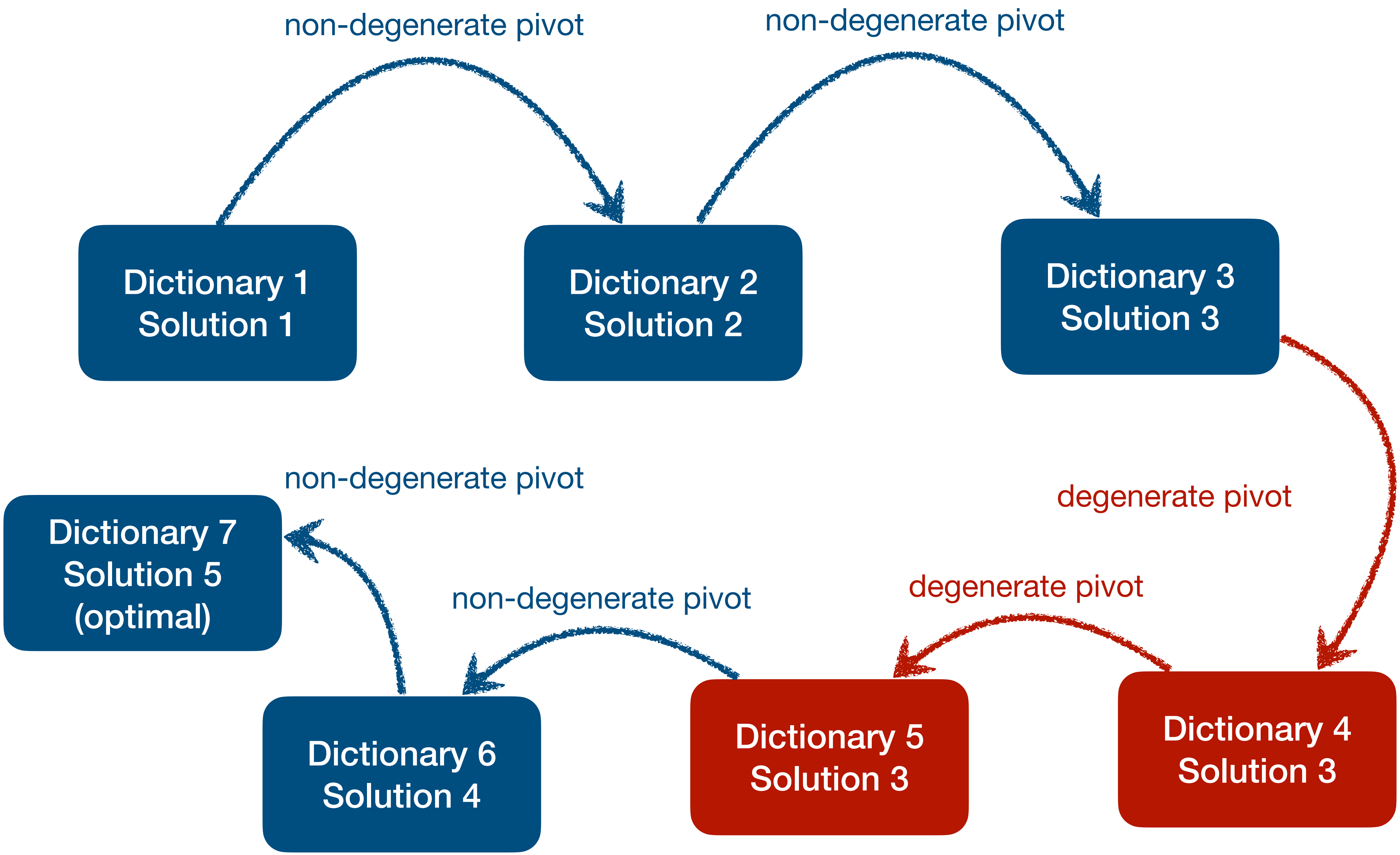
degenerate pivot

Dictionary 5
Solution 3

non-degenerate pivot

Dictionary 6
Solution 4

Pictorially



non-degenerate pivot

non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Dictionary 3
Solution 3

degenerate pivot

Dictionary 4
Solution 3

degenerate pivot

Dictionary 5
Solution 3

non-degenerate pivot

Dictionary 6
Solution 4

non-degenerate pivot

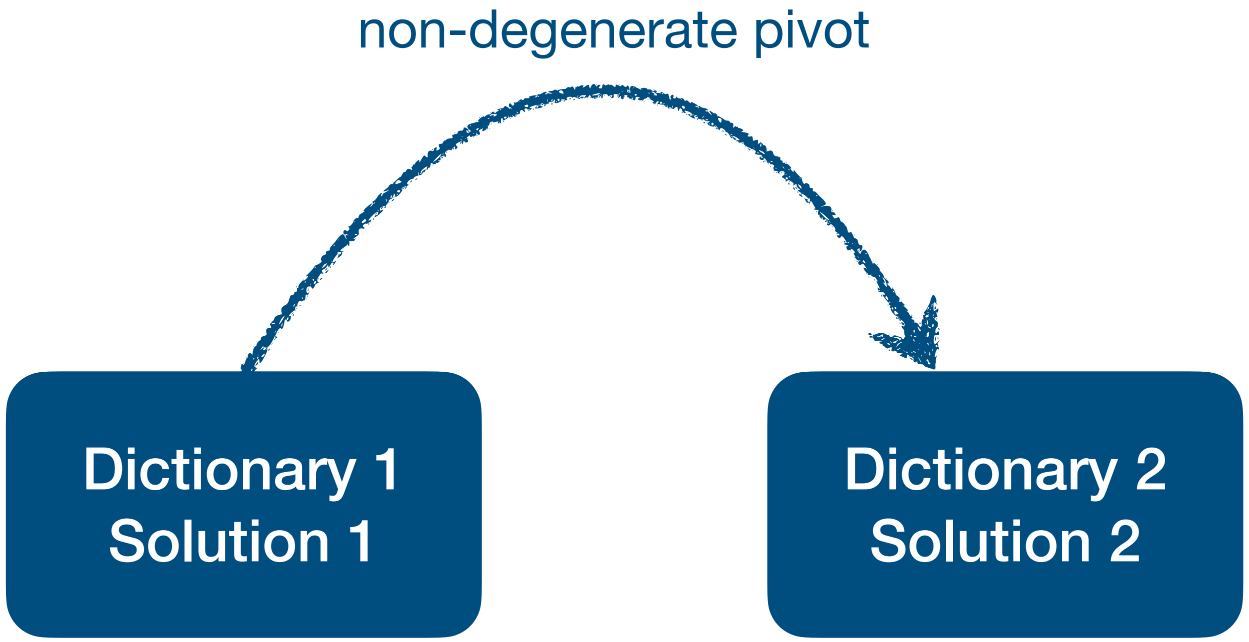
Dictionary 7
Solution 5
(optimal)

Could this happen though?

Could this happen though?

Dictionary 1
Solution 1

Could this happen though?

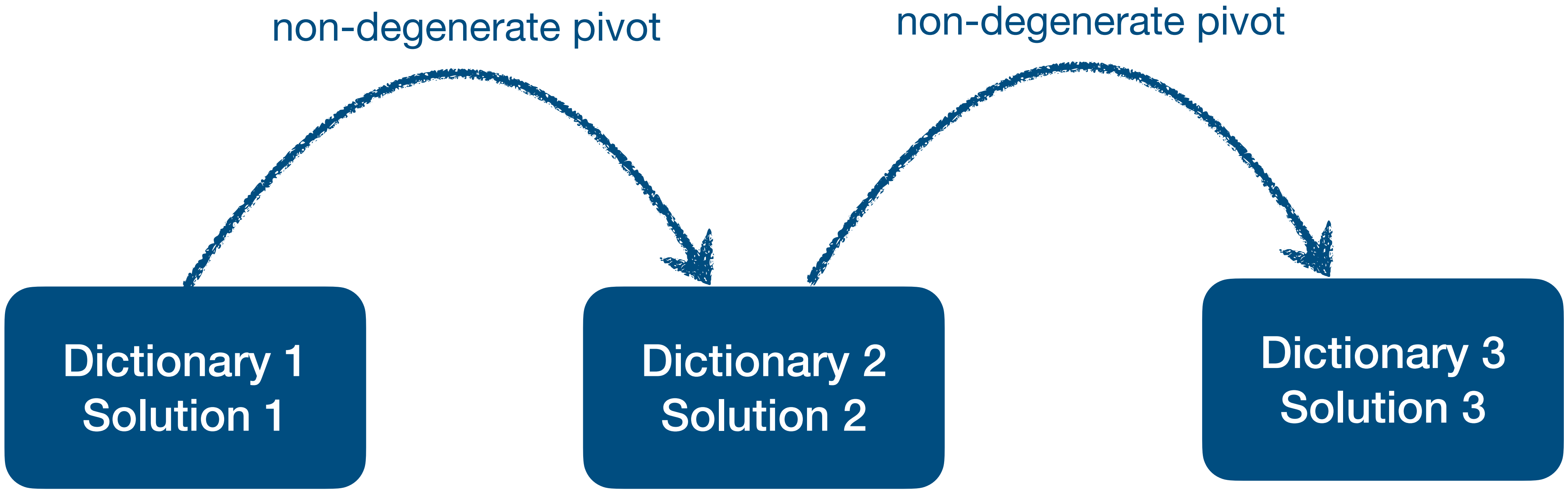


non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Could this happen though?



non-degenerate pivot

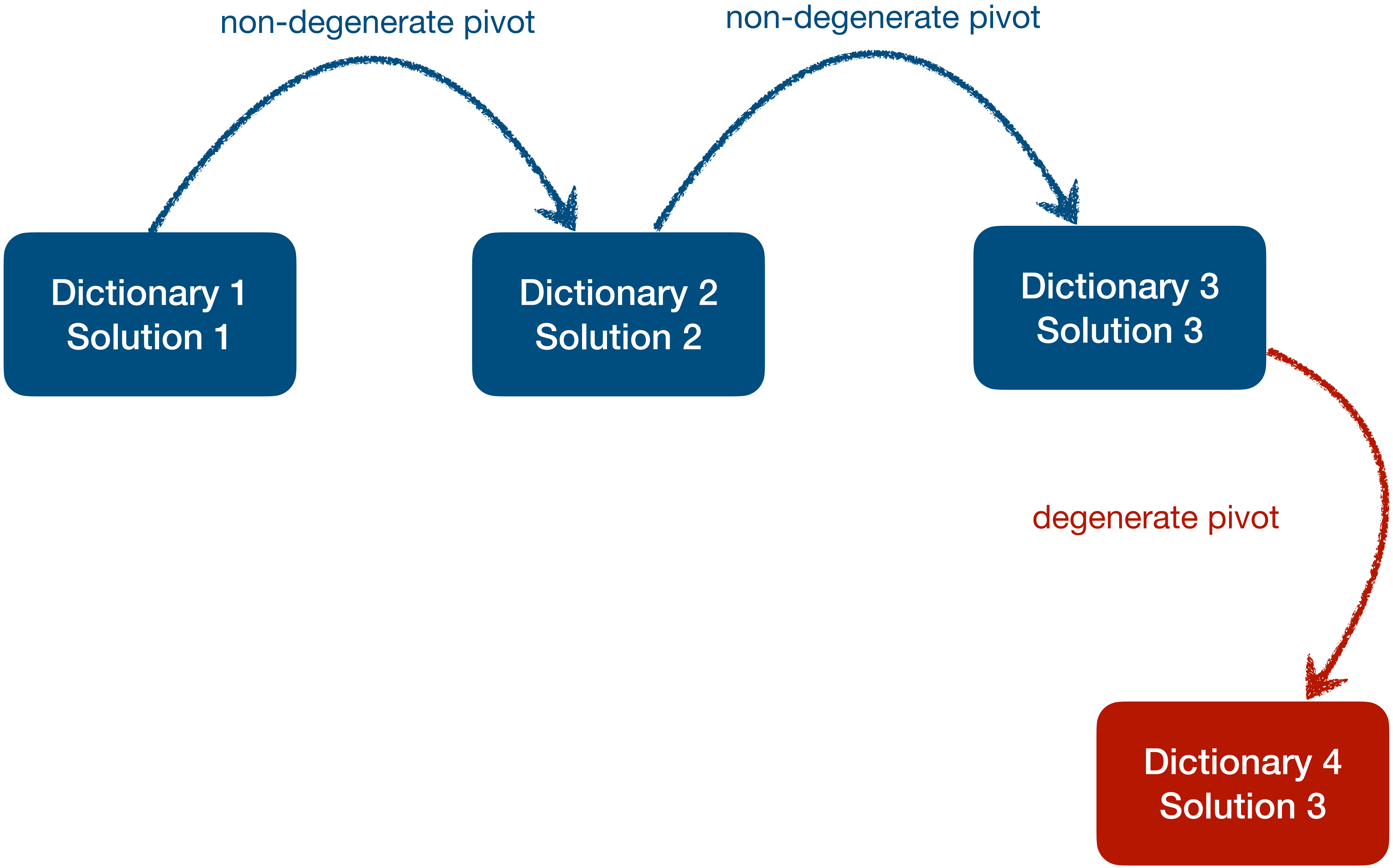
non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Dictionary 3
Solution 3

Could this happen though?



non-degenerate pivot

non-degenerate pivot

Dictionary 1
Solution 1

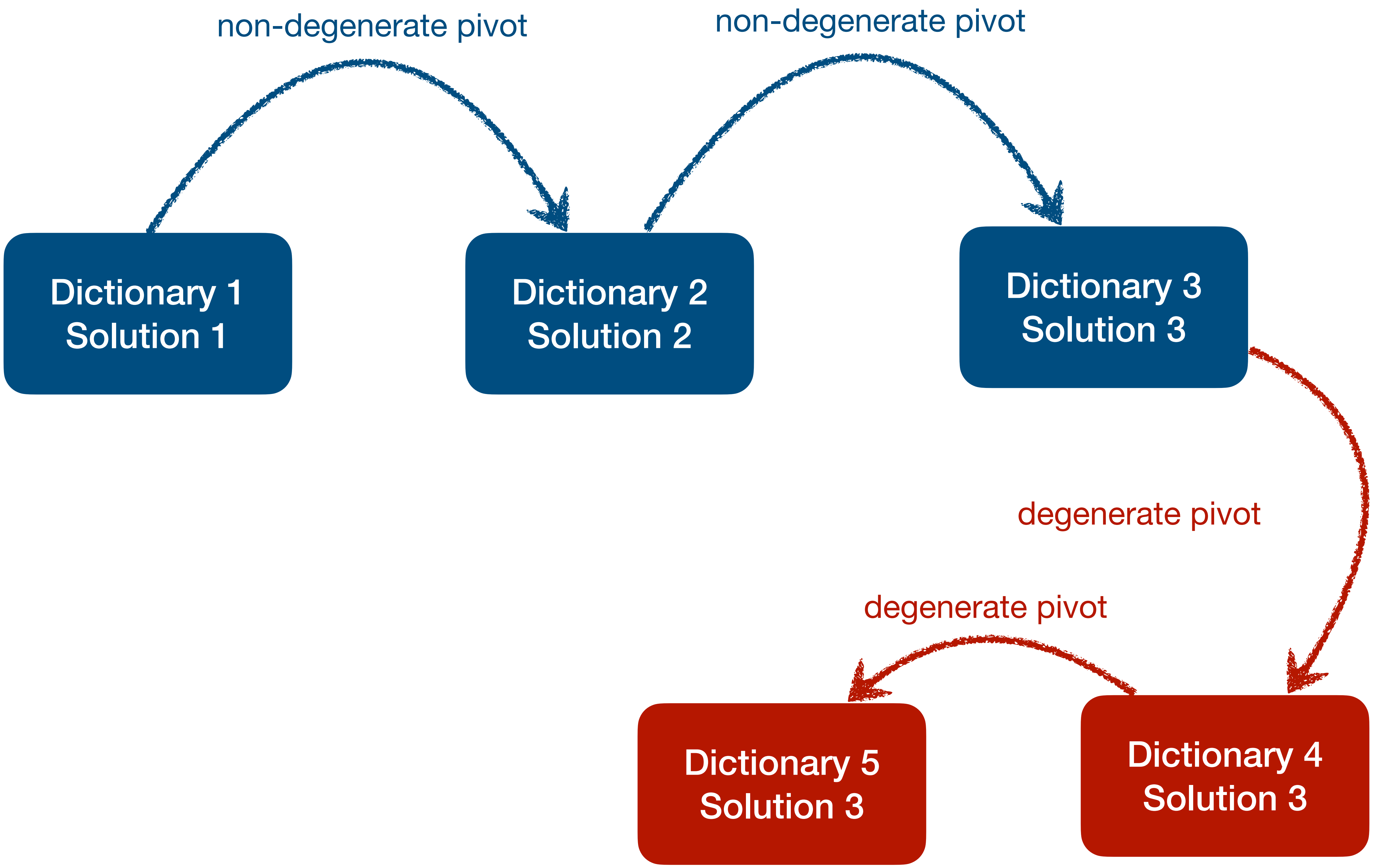
Dictionary 2
Solution 2

Dictionary 3
Solution 3

degenerate pivot

Dictionary 4
Solution 3

Could this happen though?



non-degenerate pivot

non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Dictionary 3
Solution 3

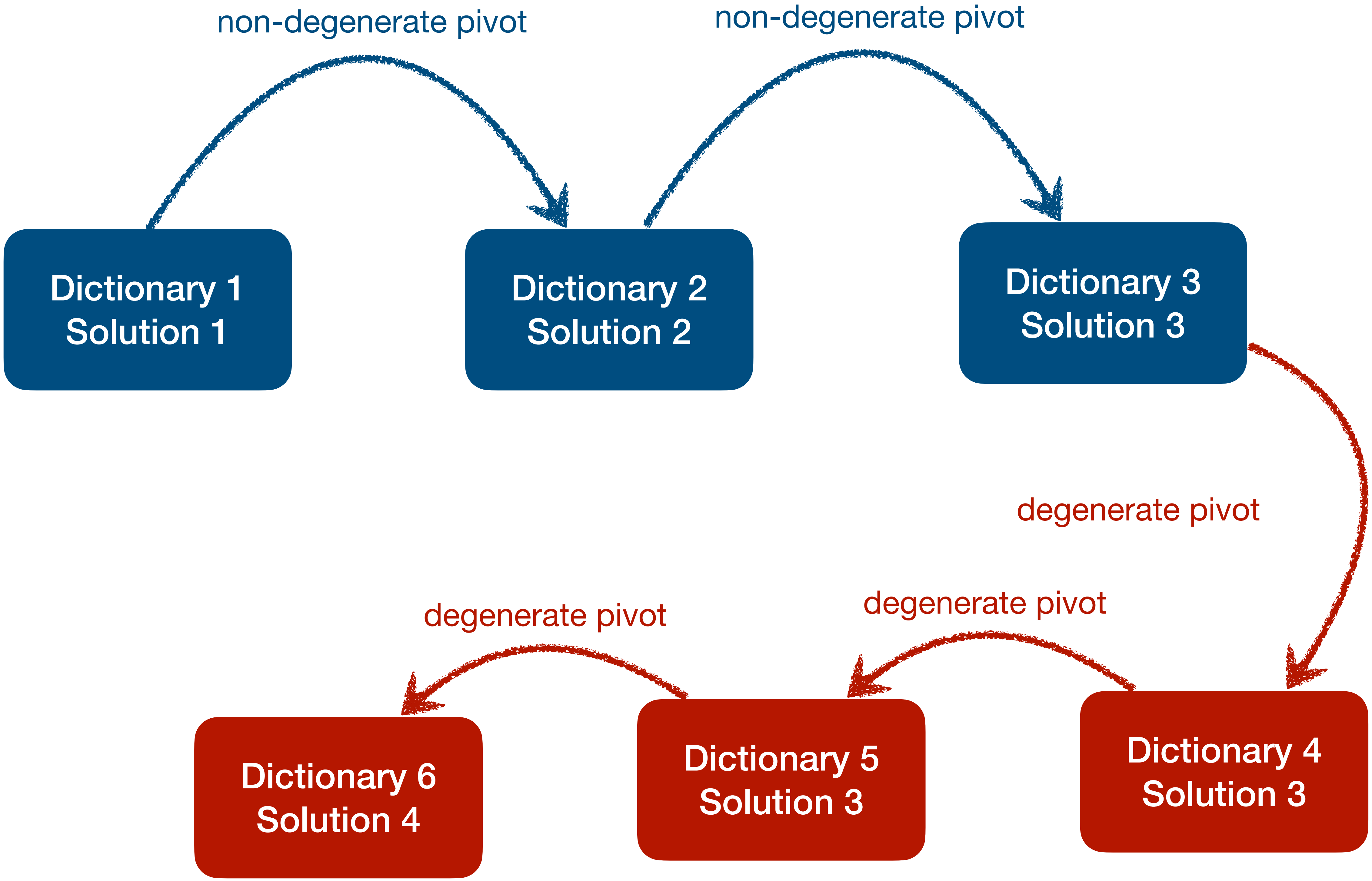
degenerate pivot

degenerate pivot

Dictionary 5
Solution 3

Dictionary 4
Solution 3

Could this happen though?



non-degenerate pivot

non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Dictionary 3
Solution 3

degenerate pivot

degenerate pivot

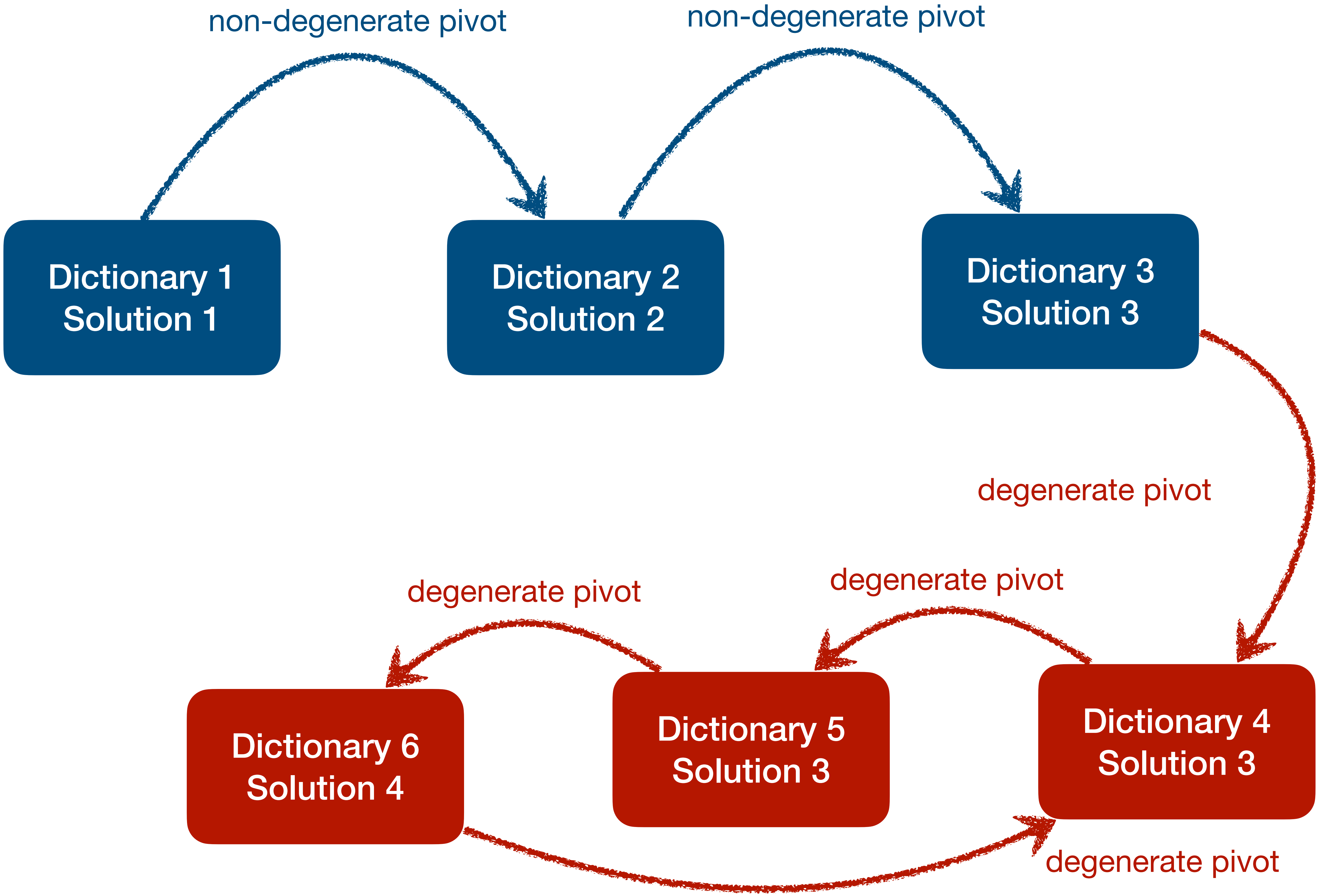
degenerate pivot

Dictionary 6
Solution 4

Dictionary 5
Solution 3

Dictionary 4
Solution 3

Could this happen though?



non-degenerate pivot

non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Dictionary 3
Solution 3

degenerate pivot

degenerate pivot

degenerate pivot

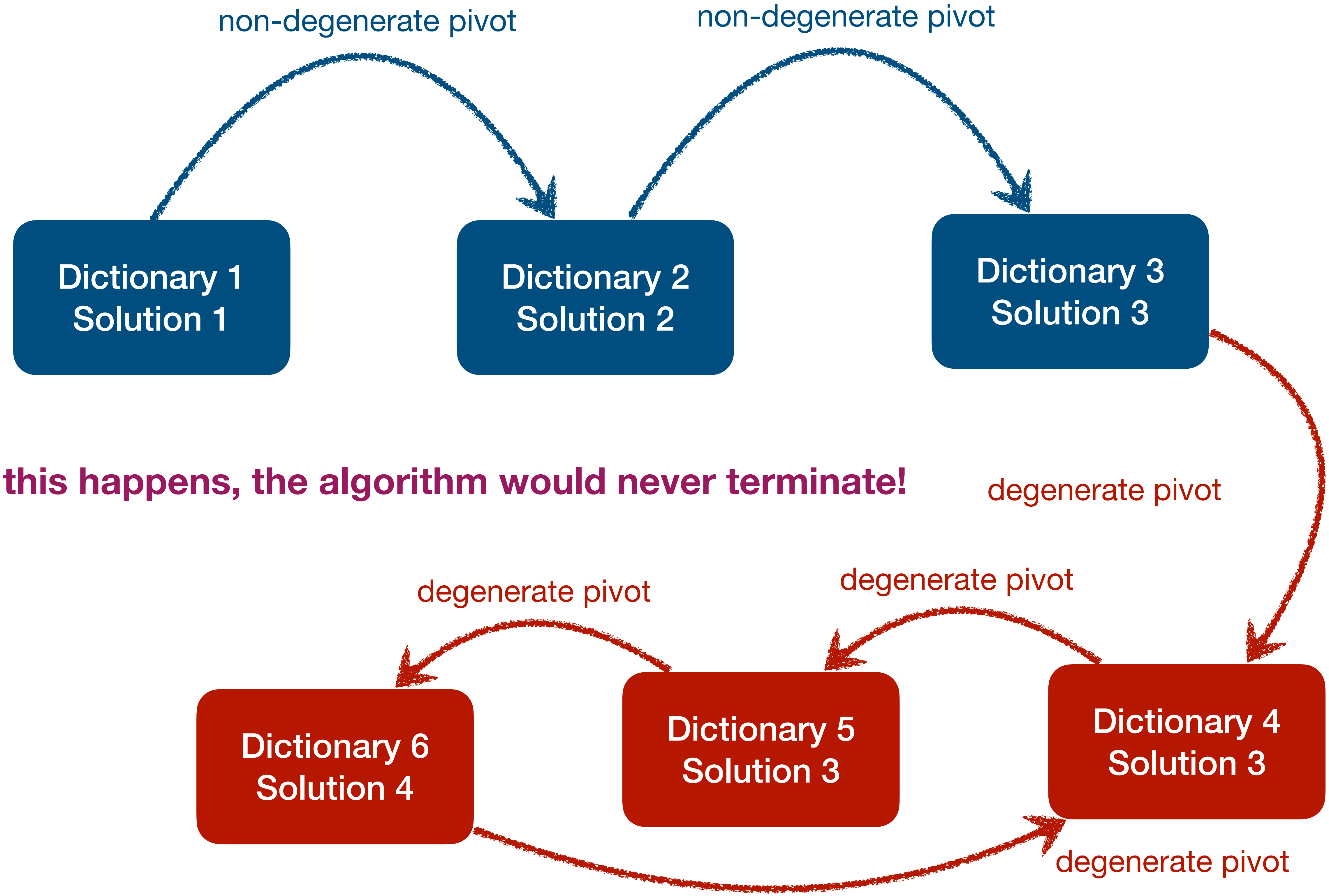
Dictionary 6
Solution 4

Dictionary 5
Solution 3

Dictionary 4
Solution 3

degenerate pivot

Could this happen though?



non-degenerate pivot

non-degenerate pivot

Dictionary 1
Solution 1

Dictionary 2
Solution 2

Dictionary 3
Solution 3

If this happens, the algorithm would never terminate!

degenerate pivot

degenerate pivot

degenerate pivot

Dictionary 6
Solution 4

Dictionary 5
Solution 3

Dictionary 4
Solution 3

degenerate pivot

Cycling

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In theory: Cycling can happen.

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In practice: Cycling rarely happens.

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But non-degenerate pivots are quite common.

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Can we avoid cycling in theory too?

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But non-degenerate pivots are quite common.

Can we avoid cycling in theory too?

Bland's rule: For both the entering variable and the leaving variable, choose the one with the smallest index.

Termination

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Proof: A dictionary is determined by which variables are basic and which are non-basic.

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Even more good news: We have other algorithms that run in worst-case polynomial running time (Ellipsoid Method, Interior Point Methods).