Algorithmic Game Theory and Applications

Nash Equilibrium and Zero-Sum Games

Solution Concept #3:
Pure Nash Equilibrium

Pure Nash Equilibrium (PNE): A pure strategy profile $(s_1, \ldots, s_n)$ such that for any player $i \in N$, *fixing the pure strategies* s_{-i} *of the other players*, player i cannot get higher utility from choosing a different pure strategy.

Mathematically: $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

Equivalently: $s_i \in \arg\max_{\hat{s}_i \in S_i} u_i(\hat{s}_i, s_{-i})$

In words: s_i is a pure strategy that maximises the utility of the player, given the fixed strategies s_{-i} of the other players.

Terminology: s_i is a *pure best response* to s_{-i}.

Terminology: Player i does not have a profitable *unilateral deviation*.

Solution Concept #3*: (Mixed) Nash Equilibrium

Pure Nash Equilibrium (MNE): A mixed strategy profile $(x_1, \ldots, x_n)$ such that for any player $i \in N$, *fixing the mixed strategies x_{-i} of the other players*, player i cannot get higher utility from choosing a different mixed strategy.

Mathematically: $u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i})$ for all $x_i' \in \Delta(S_i)$.

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(Mixed) Nash Equilibrium

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Recall:
$u_i(\hat{x}_i, x_{-i}) = \mathbb{E}_{(s_i, s_{-i}) \sim (x_i, x_{-i})}[u_i(s_i, s_{-i})]$

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Fundamental Proposition

<u>Proposition 1:</u> A mixed strategy profile $x = (x_i, x_{-i})$ is a mixed Nash Equilibrium (MNE) if and only if, for every player $i \in N$ and every pure strategy $s_i' \in S_i$, we have

$$u_i(x_i, x_{-i}) \geq u_i(s_i', x_{-i})$$

Rock-Paper-Scissors

Consider the symmetric strategy (R, P, S) = (1/3, 1/3, 1/3) for both players. This is a MNE.

	Rock	Paper	Scissors	
Rock (R)	0, 0	-1, 1	1, -1	**1/3**
Paper (P)	1, -1	0, 0	-1, 1	**1/3**
Scissors (S)	-1, 1	1, -1	0, 0	**1/3**
	1/3	**1/3**	**1/3**	

$u_1(R, x_2) = 0$ →

$u_1(P, x_2) = 0$ →

$u_1(S, x_2) = 0$ →

$$u_1(x_1, x_2) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (-1) = 0$$

Quick Recap: Efficient Algorithms

$O(\log n)$	$O(n)$	$O(n \log n)$	$O(n^2)$	$O(n^\alpha)$	$O(c^n)$
logarithmic	linear		quadratic	polynomial	exponential
The algorithm does not even read the whole input.	The algorithm accesses the input only a constant number of times.	The algorithm splits the inputs into two pieces of similar size, solves each part and merges the solutions.	The algorithm considers pairs of elements.	The algorithm performs many nested loops.	The algorithm considers many subsets of the input elements.

constant	$O(1)$	superlinear	$\omega(n)$
superconstant	$\omega(1)$	superpolynomial	$\omega(n^\alpha)$
sublinear	$o(n)$	subexponential	$o(c^n)$

Quick Recap: Efficient Algorithms

Polynomial time

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Before we talk about efficient algorithms, we need to be sure about what our input is.

An efficient algorithm for verifying Nash equilibria

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Informally:

Input: A game in normal form, a mixed strategy profile $x = (x_1, \ldots, x_n)$.

Output: Yes if x is a MNE and No if it is not.

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Formally:

Input: The number n of players, the pure strategy sets S_i, given explicitly, by listing all of their elements, the utility functions u_i given explicitly as a list of *rational* numbers, one for each pure strategy profile, e.g., $u_i(s_1, \ldots, s_n)$, the mixed strategies x_i, given as vectors $(x_{i1}, \ldots, x_{im})$ of *rational* numbers, where $m = |S_i|$.

Output: Yes if x is a MNE and No if it is not.

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$$\sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} x_1(s_1) \cdot x_2(s_2) \cdot \ldots \cdot x_n(s_n) \cdot u_i(s_1, \ldots, s_n)$$

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$$u_i(x_i, x_{-i}) \geq u_i(s_i', x_{-i})$$

Another Fundamental Proposition

Proposition 2: A mixed strategy profile $x = (x_i, x_{-i})$ is a mixed Nash Equilibrium (MNE) if and only if, for every player $i \in N$, and for every pure strategy $s_i \in S_i$ in the support of x_i (i.e., $x_i(s_i) > 0$), we have $u_i(x_i, x_{-i}) = u_i(s_i, x_{-i})$.

A quick proof of $\Leftarrow$

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Let $x = (x_i, x_{-i})$ be a MNE. This immediately implies $u_i(s_i, x_{-i}) \leq u_i(x_i, x_{-i})$ for all $s_i \in S_i$.

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Assume by contradiction that for some strategy $s_i' \in \text{supp}(x_i)$, we had $u_i(s_i, x_{-i}) < u_i(x_i, x_{-i})$

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Let $s_i^* \in \arg\max_{s_i \in \text{supp}(x_i)} u_i(s_i, x_{-i})$. By definition this is $\geq u_i(x_i, x_{-i})$

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Consider the alternative mixed strategy x_i' such that $x_i'(s_i) = x_i(s_i)$ for all pure strategies $s_i \neq s_i', s_i^*$ and

$x_i(s_i') = 0$

$x_i'(s_i^*) = x_i(s_i^*) + x_i(s_i')$

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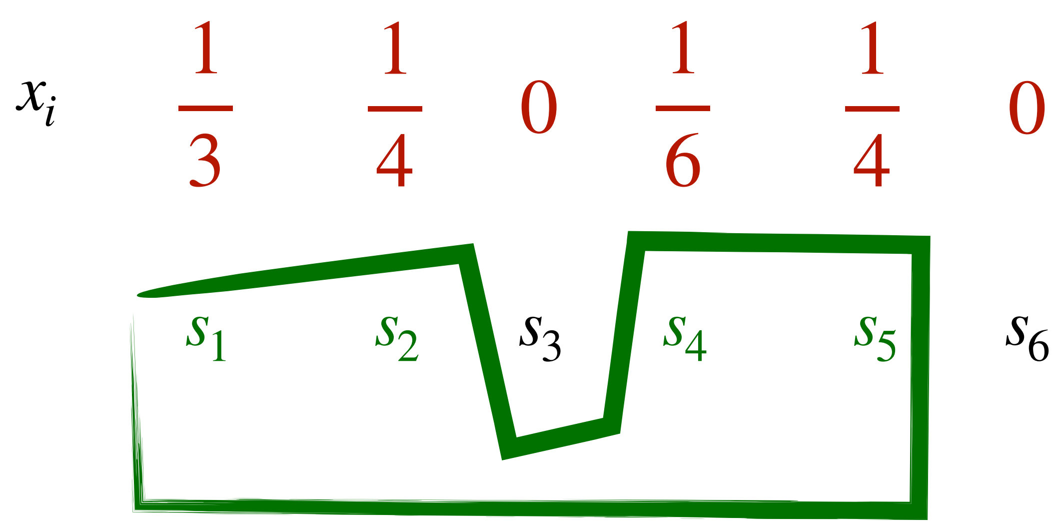
x_i' results in higher utility, a contradiction!

Via example:

x_i	$\frac{1}{3}$	$\frac{1}{4}$	0	$\frac{1}{6}$	$\frac{1}{4}$	0
	s_1	s_2	s_3	s_4	s_5	s_6

Via example:



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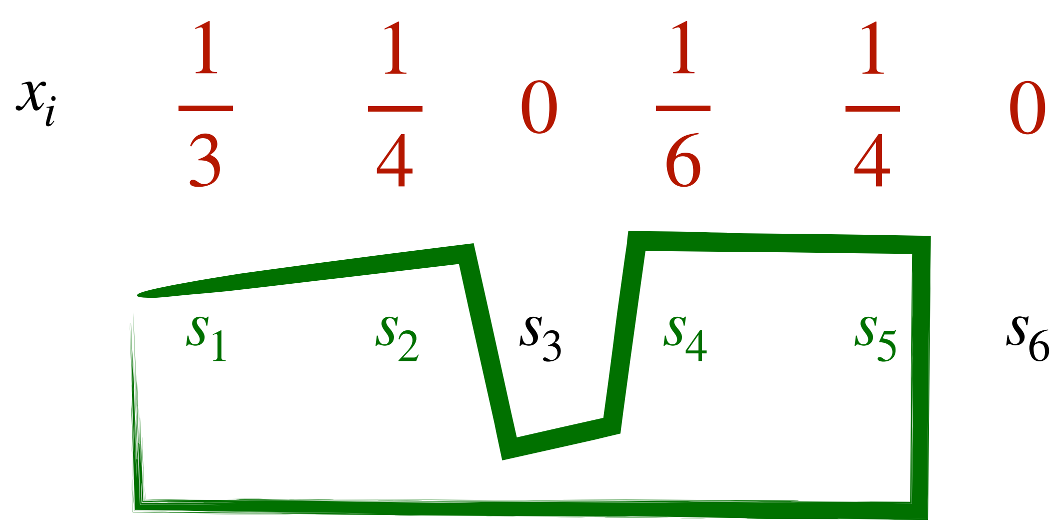

Claim: The utility $u_i(s_i, x_{-i})$ for every strategy in the support is the same.

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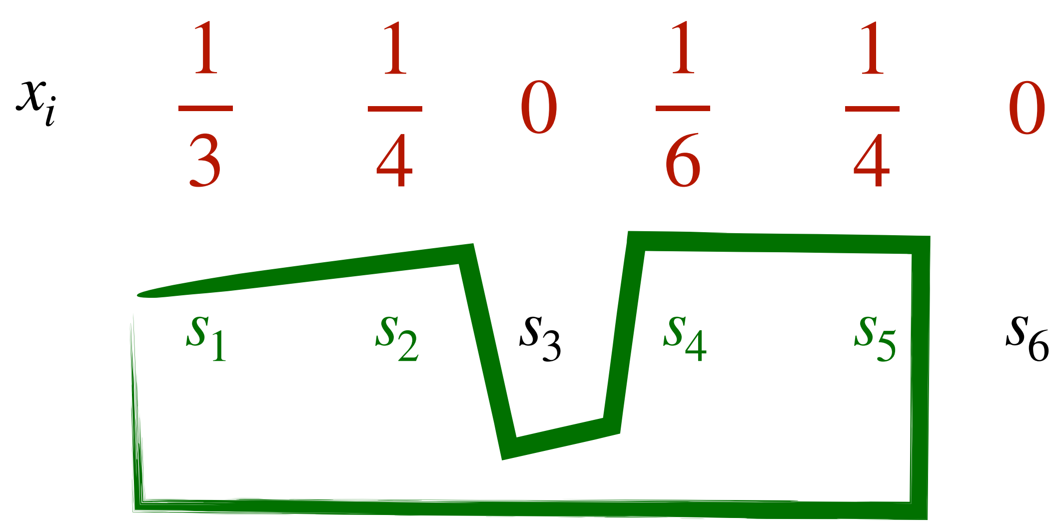

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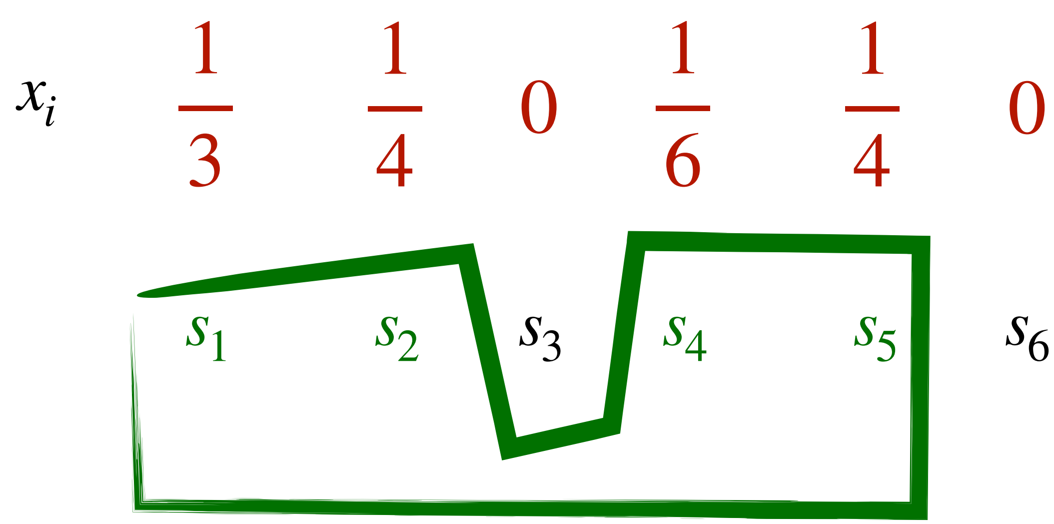

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By contradiction: Assume that this is not the case.

Then there are two pure strategies s_i, s_j such that s_i gives less utility than s_j.

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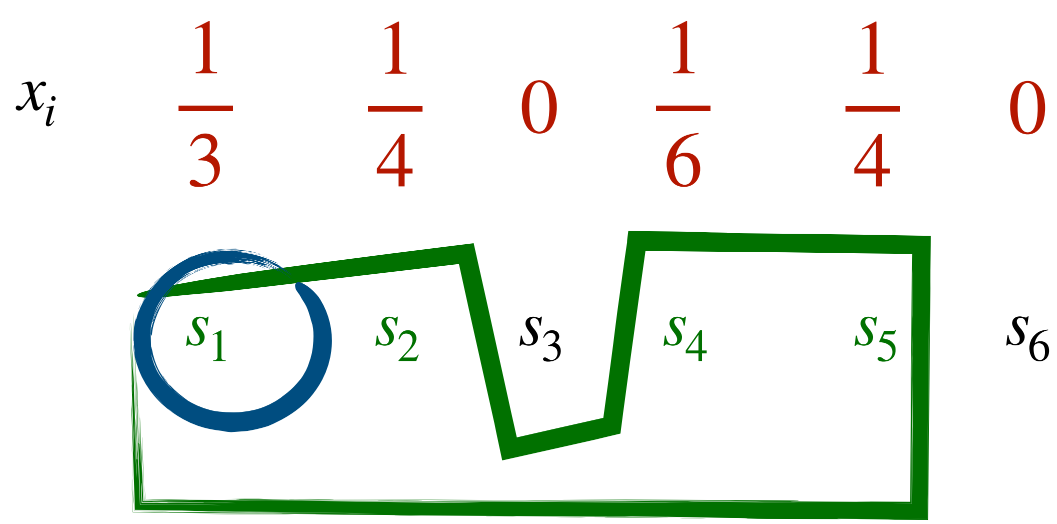

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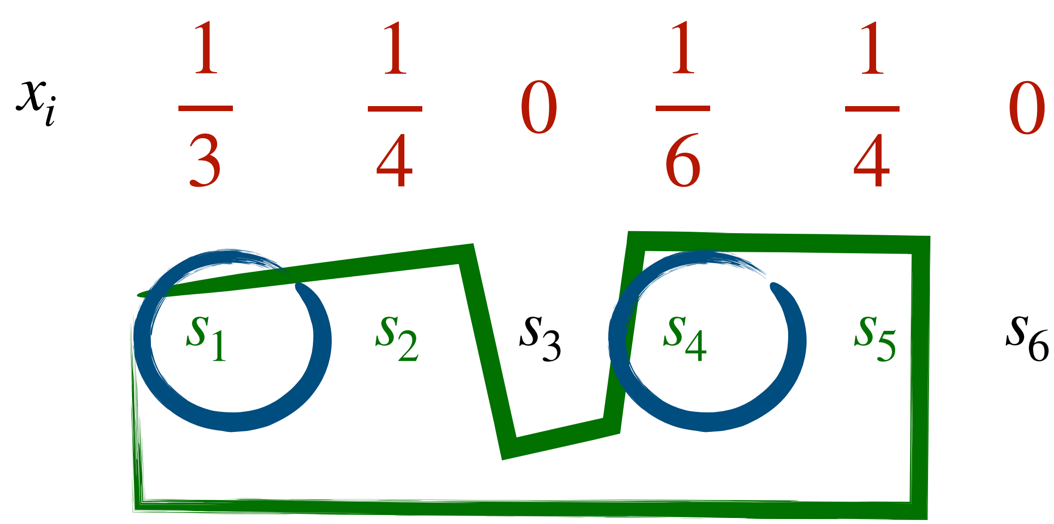

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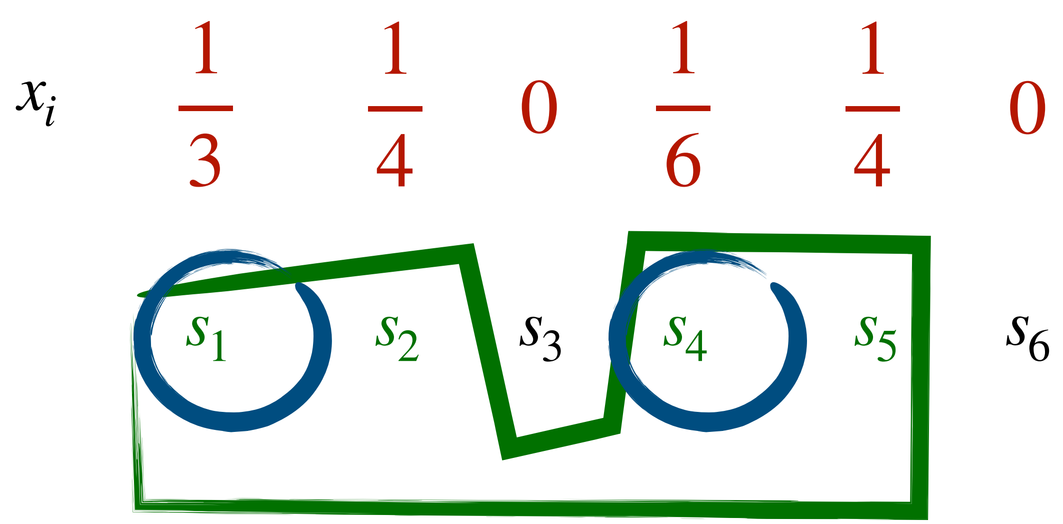

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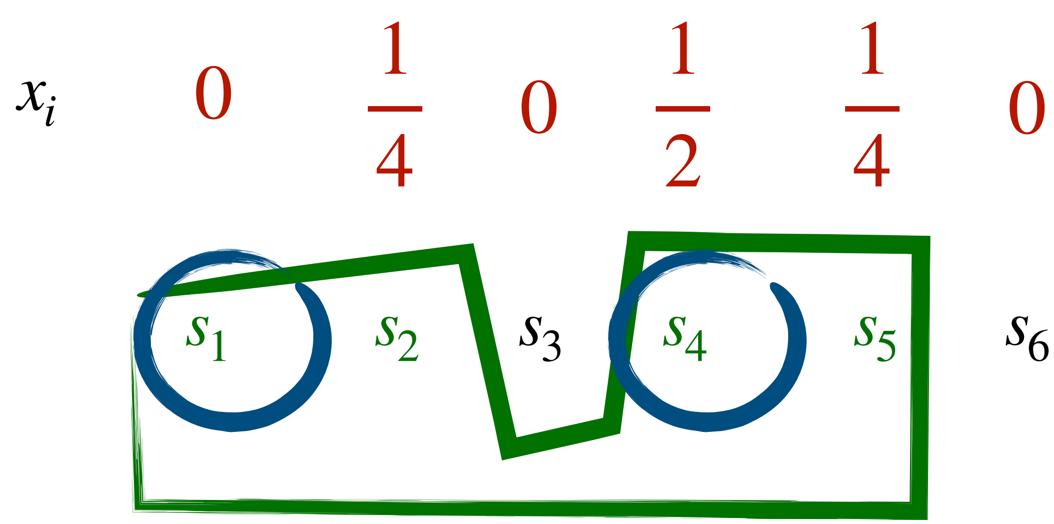

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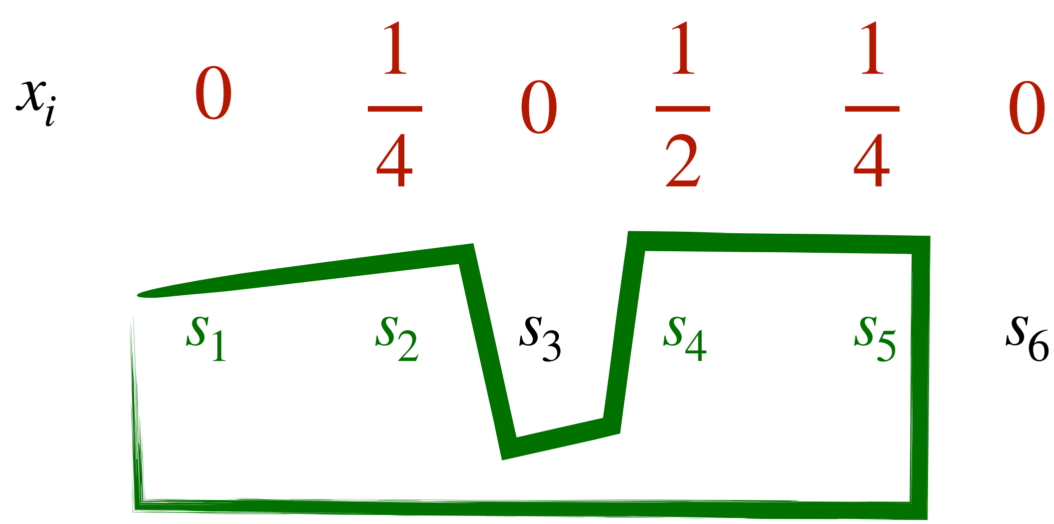

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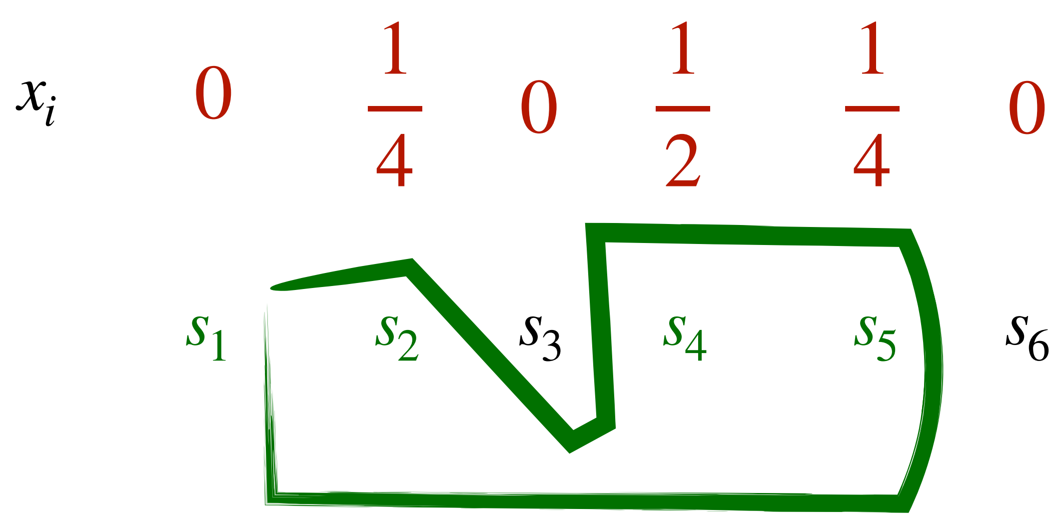

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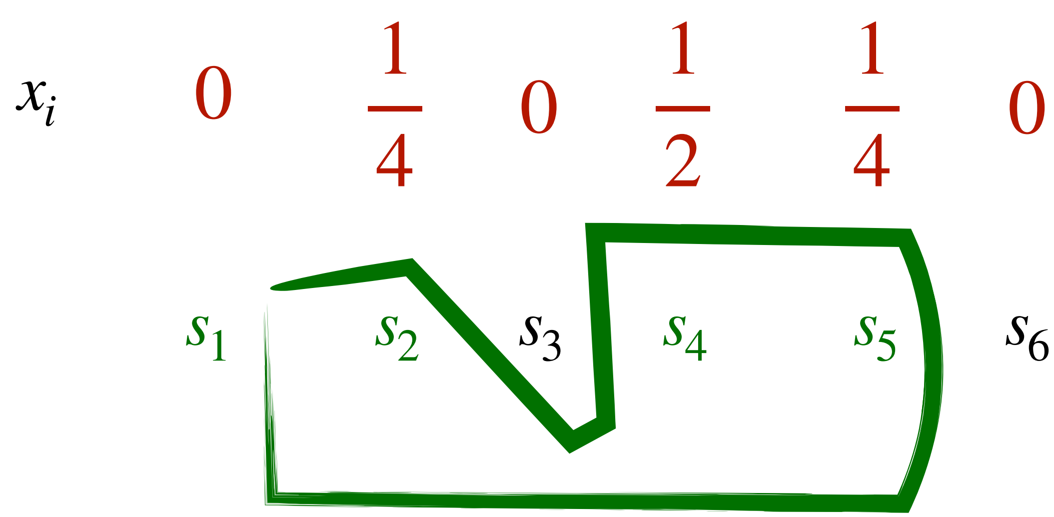

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Take the probability from s_i and move it to s_j.

We have created a better (i.e., with higher expected utility) mixed strategy x_i'.

Back to Choosing the TV show

Both
(Peep Show, Peep Show)
and
(FOTC, FOTC),
are PNE!

	Peep Show	FOTC
Peep Show	10, 7	5, 5
FOTC	1, 1	7, 10

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Remember: By definition, PNE are MNE, so we already have found two!

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We call those *fully mixed*.

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Assume that we have a mixed equilibrium (x, y)

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<u>Note:</u> We use x and y here instead of x_1 and x_2 because we only have two players. We will therefore use x_i, y_i to denote probabilities.

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Let (x_1, x_2) be the mixed strategy of Player 1 and (y_1, y_2) be the mixed strategy of Player 2.

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Back to Choosing the TV show

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By Proposition 2, we know that

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Another Fundamental Proposition

Proposition 2: A mixed strategy profile $x = (x_i, x_{-i})$ is a mixed Nash Equilibrium (MNE) if and only if, for every player $i \in N$, and for every pure strategy $s_i \in S_i$ in the support of x_i (i.e., $x_i(s_i) > 0$), we have $u_i(x_i, x_{-i}) = u_i(s_i, x_{-i})$.

Question: *Can you translate the idea we just used into an algorithm, which takes advantage of the proposition above?*

An algorithm for computing Nash equilibria in 2-player games

Assume that we have *magical access* to the supports for all mixed strategies in the MNE.

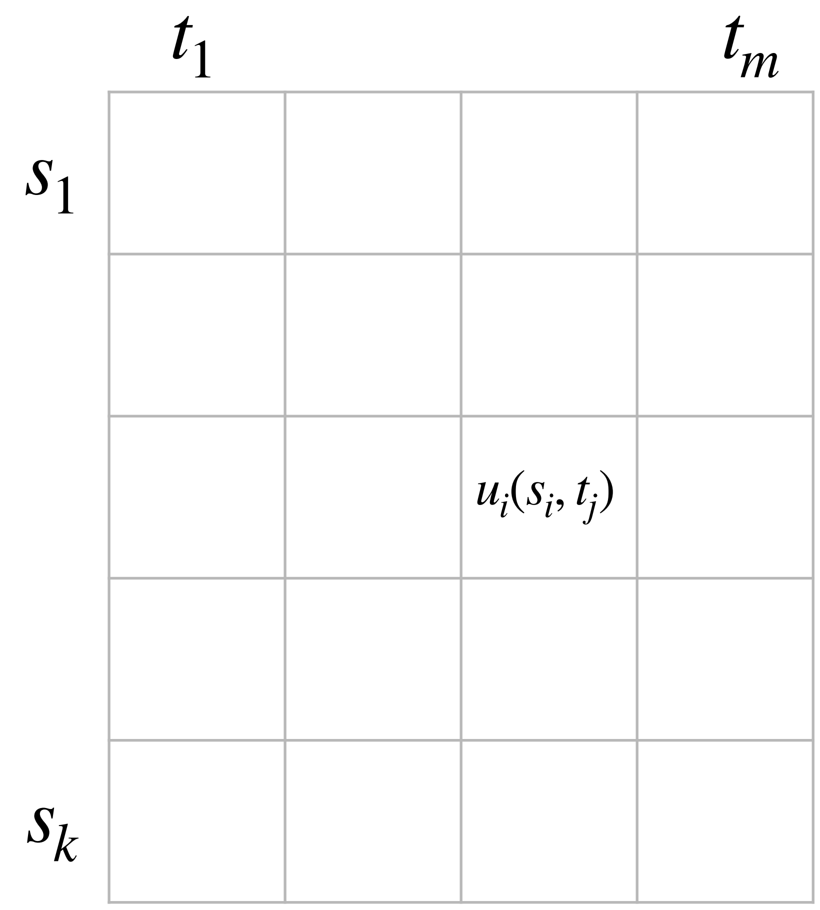
In algorithms, we often call this *oracle access*.

We can then write a set of inequalities:

$$\sum_{y_j \in \text{supp}(y)} y(t_j) \cdot u_i(s_i, t_j) \quad =$$

$$\sum_{y_j \in \text{supp}(y)} y(t_j) \cdot u_i(s_i', t_j) \text{ for all } s_i, s_i \in \text{supp}(x)$$

$$\sum_{y_j \in \text{supp}(y)} y(t_j) = 1$$



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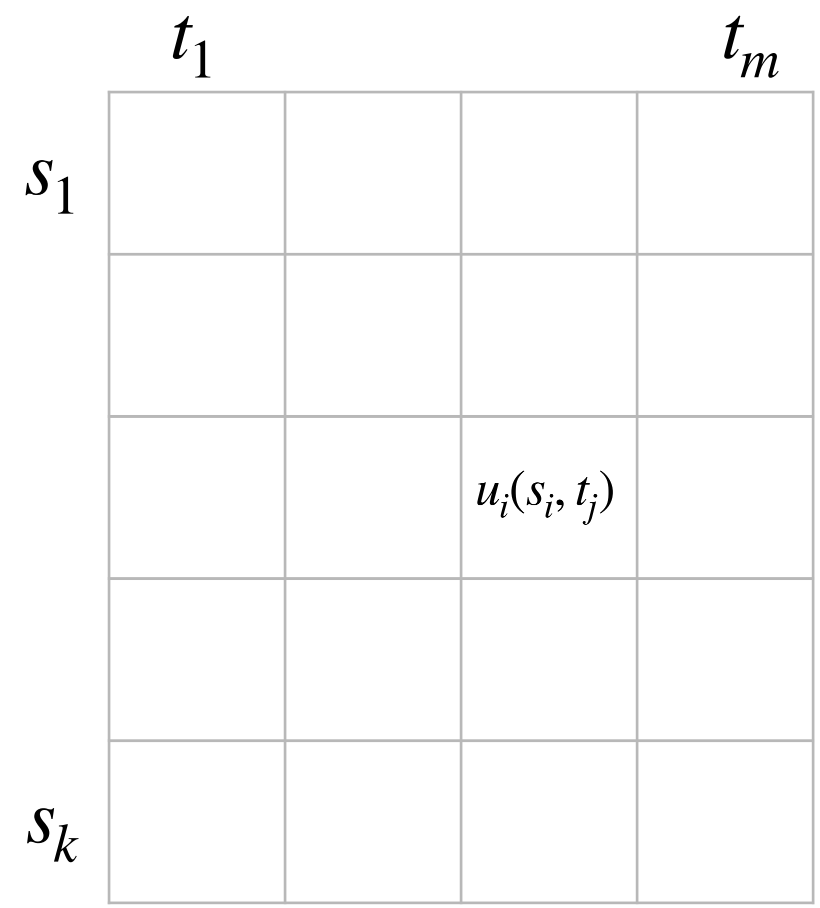

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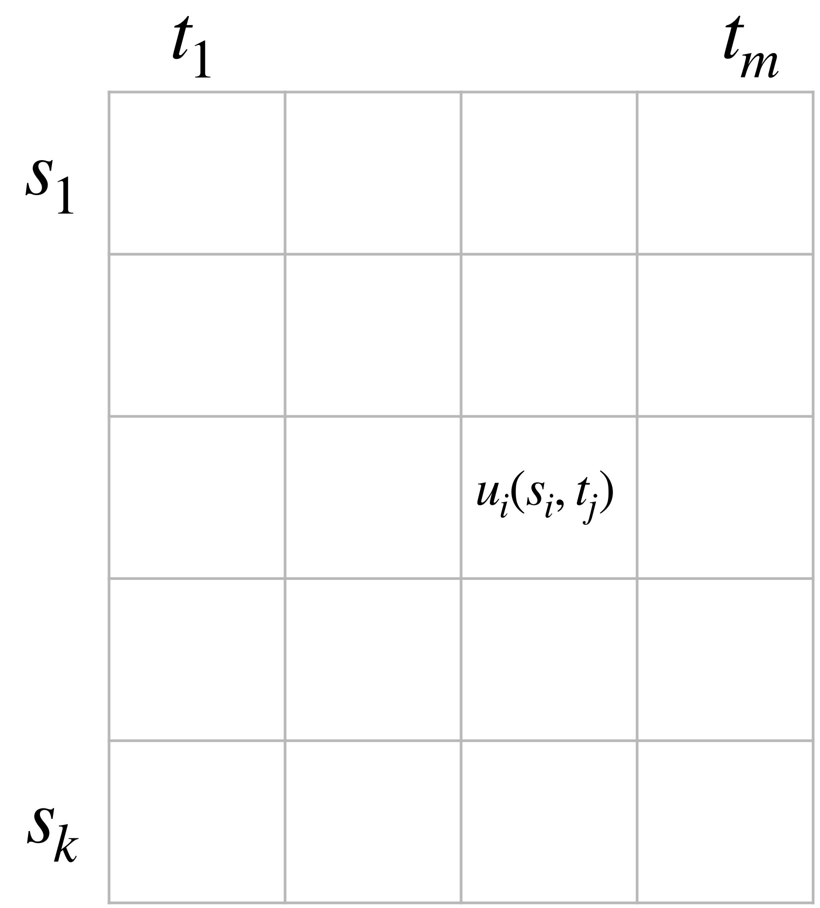

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Can we solve this in polynomial time?

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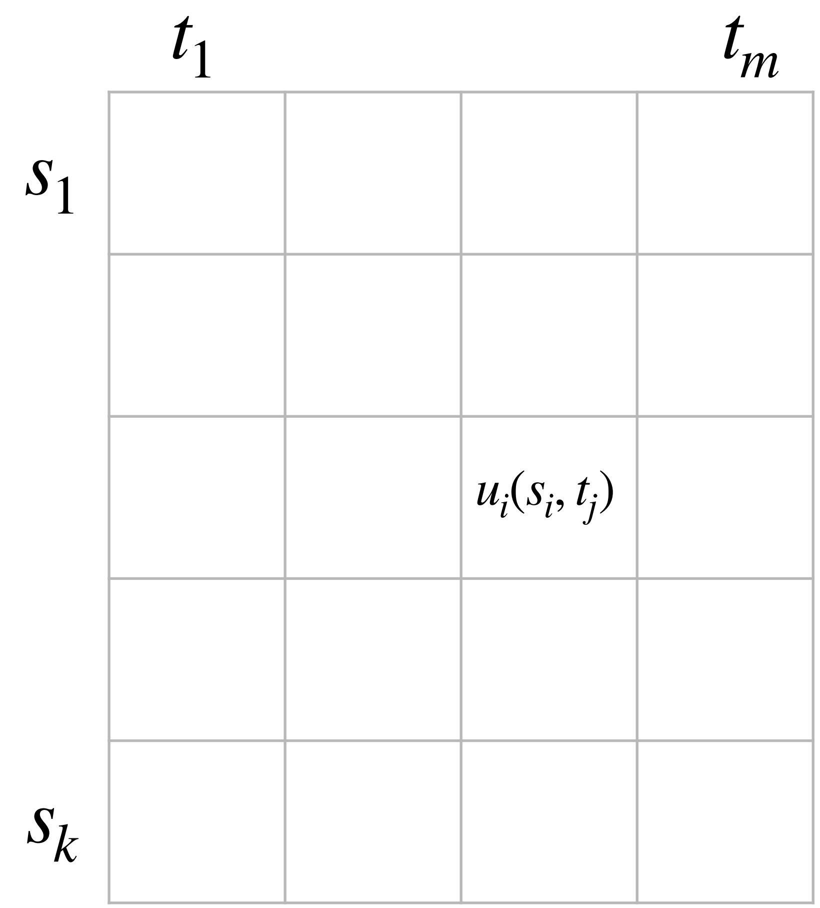

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A bit more precisely

By using the notation of utilities, we want a solution to the following system of inequalities:

1. $\forall i \in N, \forall s_j \in \text{ supp}(x_i), \quad u_i(s_j, x_{-i}) = w_i$ (Proposition 2)
2. $\forall i \in N, \forall s_j \notin \text{supp}(x_i), \quad u_i(s_j, x_{-i}) \leq w_i$ (MNE condition)
3. $\forall i \in N, \quad \sum_{s_j \in S_j} x_i(s_j) = 1$ (probabilities)
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This actually holds for any number of players, but the inequalities are linear only for two players. Why?

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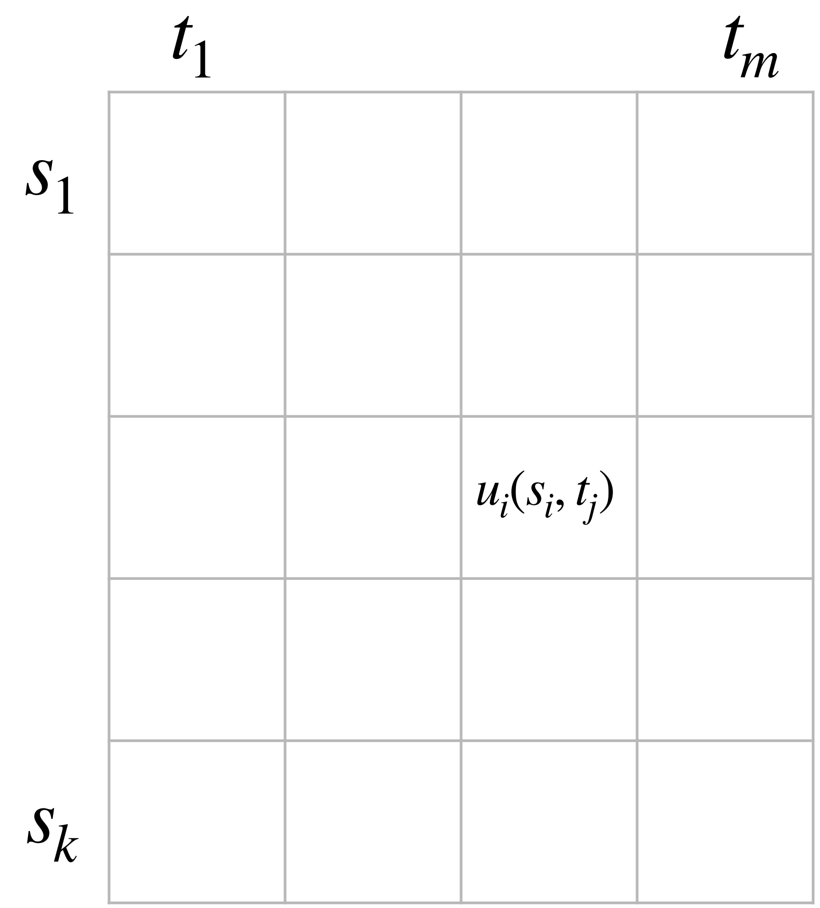

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What about this?

	t_1			t_m
s_1				
			$u_i(s_i, t_j)$	
s_k				

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Question 2: How fast is this algorithm? How many possible supports are there?

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2 players with pure strategy sets

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Player 1 is trying to maximise the utility (maximiser) and Player 2 is trying to minimise it (minimiser).

Rock-Paper-Scissors

	Rock	Paper	Scissors
Rock (R)	0, 0	-1, 1	1, -1
Paper (P)	1, -1	0, 0	-1, 1
Scissors (S)	-1, 1	1, -1	0, 0

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Why is this the rational thing to do in Zero-Sum games?

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von Neumann's Minimax Theorem (1928, 1944): $v_x = v_y$

von Neumann’s Theorem

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- If the maximiser played a strategy that could only achieve a smaller payoff, or the minimiser played a strategy that incurred a higher loss, they could switch to the minimax/maximin (optimal) strategies.

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- So these strategies are the only reasonable/rational outcomes of the game.

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Aren’t MNE reasonable outcomes of games?

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We in fact computed an MNE is RPS and it looks pretty reasonable!

Rock-Paper-Scissors

Consider the symmetric strategy (R, P, S) = (1/3, 1/3, 1/3) for both players. These are **optimal strategies.**

	Rock	Paper	Scissors	
Rock (R) $u_1(R, x_2) = 0$ →	0, 0	-1, 1	1, -1	**1/3**
Paper (P) $u_1(P, x_2) = 0$ →	1, -1	0, 0	-1, 1	**1/3**
Scissors (S) $u_1(S, x_2) = 0$ →	-1, 1	1, -1	0, 0	**1/3**
	1/3	**1/3**	**1/3**	

$$u_1(x_1, x_2) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (-1) = 0$$

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Theorem: Let (x^*, y^*) be a pair of mixed strategies of a 2-player Zero-Sum game. Then x^* and y^* are both optimal strategies if and only if (x^*, y^*) is a MNE.

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The argument for the minimiser is similar.

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By definition, that means that for the maximiser, we have

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By the minimax theorem, we know that the RHS of both (1) and (2) are equal. This is only possible if the two inequalities are satisfied with equality ⇒ both strategies are optimal.

In 2-player Zero-Sum Games

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This provides a proof of the minimax theorem. How?

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How do we solve those systems of linear equations to run the support enumeration algorithms for computing MNE?

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LINEAR PROGRAMMING

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