Algorithms and Data Structures

Degeneracy, Geometry, and Duality

Historic Note

The Simplex Method was invented by George Dantzig in 1947.

It is still being used today in most of the LP-solvers.

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> The origins of the simplex method go back to one of two famous unsolved problems in mathematical statistics proposed by Jerzy Neyman, which I mistakenly solved as a homework problem; it later

Dantzig. Origins of the Simplex Method. In *A History of Scientific Computing*, 1990.

What if we have this dictionary?

Maximise $\zeta = 5 + x_3 - x_1$

subject to

$$
\begin{aligned}
x_2 &= 5 + 2x_3 - 3x_1 \\
x_4 &= 7 - 4x_1 \\
x_5 &= x_1
\end{aligned}
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$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

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Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

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 & x_5 = & & & & & x_1 \\
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$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

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entering variable

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The LP is unbounded!

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What about this dictionary?

Maximise $\zeta = 3 - 0.5x_1 + 2x_2 - 1.5w_1$

subject to

$$x_3 = 1 - 0.5x_1 - 0.5w_1$$
$$w_2 = x_1 - x_2 + w_1$$

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x_2 cannot be increased! Are we stuck? **Degeneracy!**

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Equivalently: In a basic feasible solution, one of the basic variables is 0.

Degeneracy not necessarily and issue

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Degenerate Pivot: The entering variable stays at 0 without increasing.

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Degenerate Pivot: The entering variable stays at 0 without increasing.

"Degenerate pivots are quite common and usually harmless."

Let's not give up

Maximise $\zeta = 3 - 0.5\,x_1 + 2\,x_2 - 1.5\,w_1$

subject to

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\begin{aligned}
x_3 &= 1 - 0.5\,x_1 - 0.5\,w_1 \\
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Increase the variable as much as we can (as before): here 0 increase

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Actually pivot!

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$(x_1, x_2, x_3, w_1, w_2) = (0,0,1,0,0)$

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The new dictionary

Maximise $\zeta = 3 + 1.5\, x_1 + 2\, w_2 + 0.5\, w_1$

subject to

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\begin{aligned}
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We can now increase x_1 to $x_1 = 2$

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The pivot is not degenerate!

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It will actually lead to a final dictionary, and an optimal solution.

Pictorially

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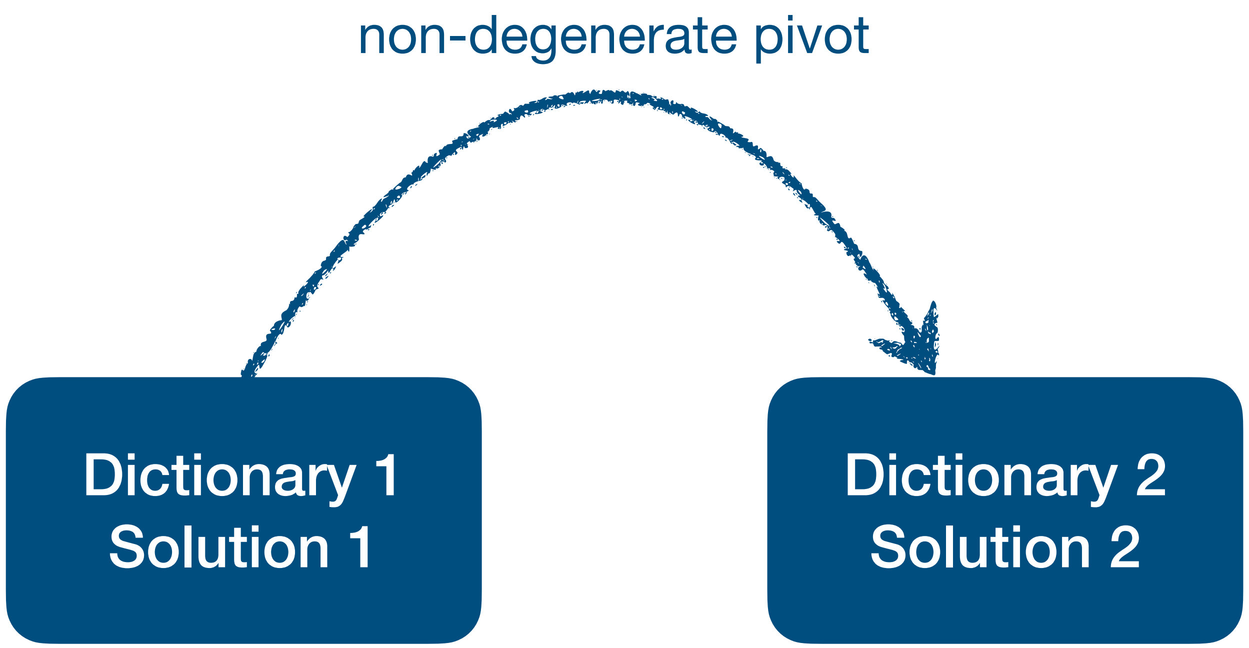
Dictionary 1
Solution 1

Pictorially



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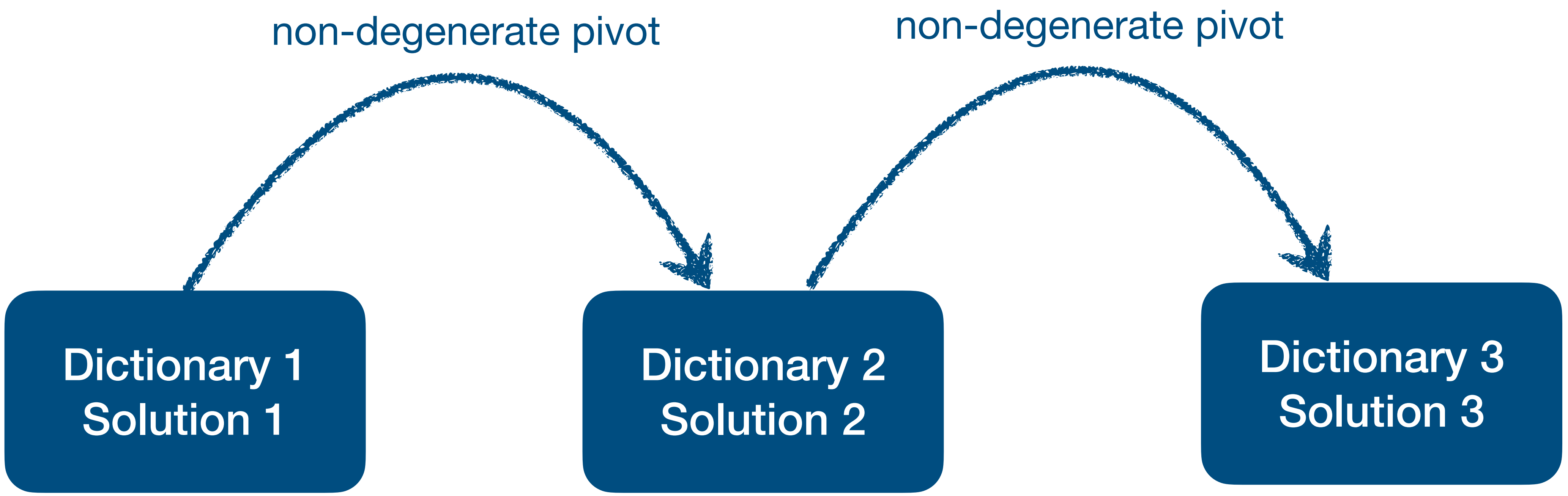

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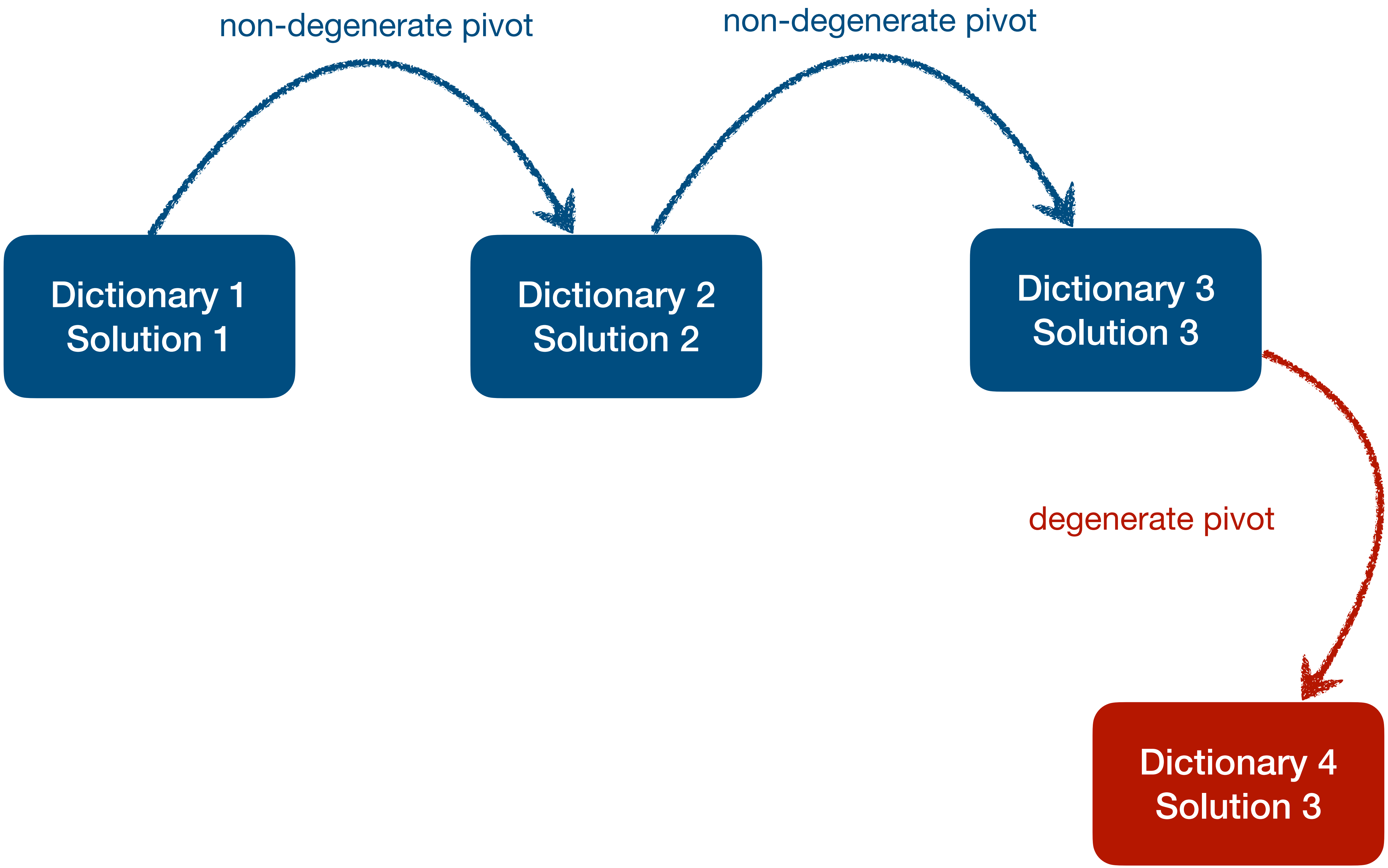

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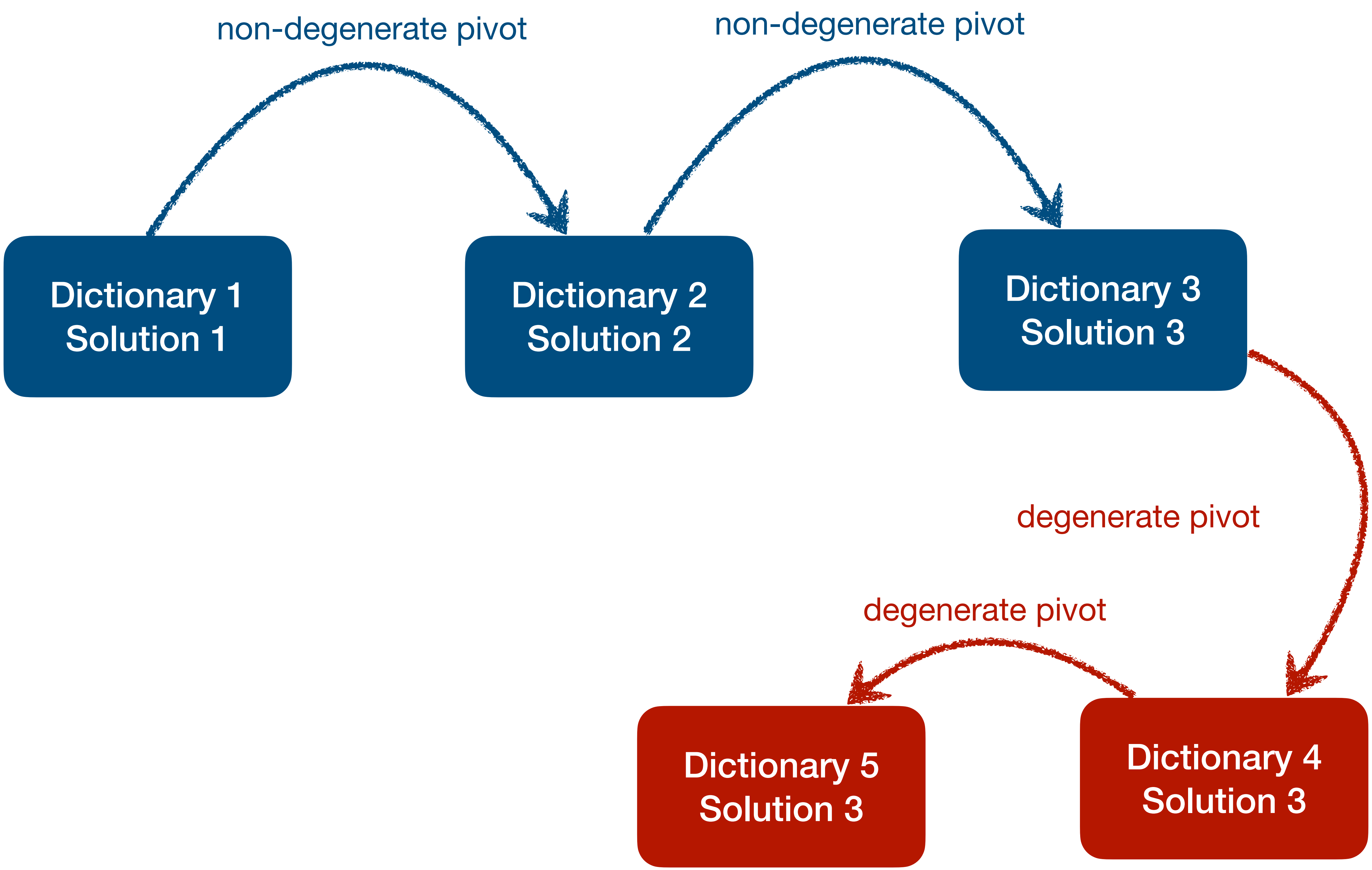

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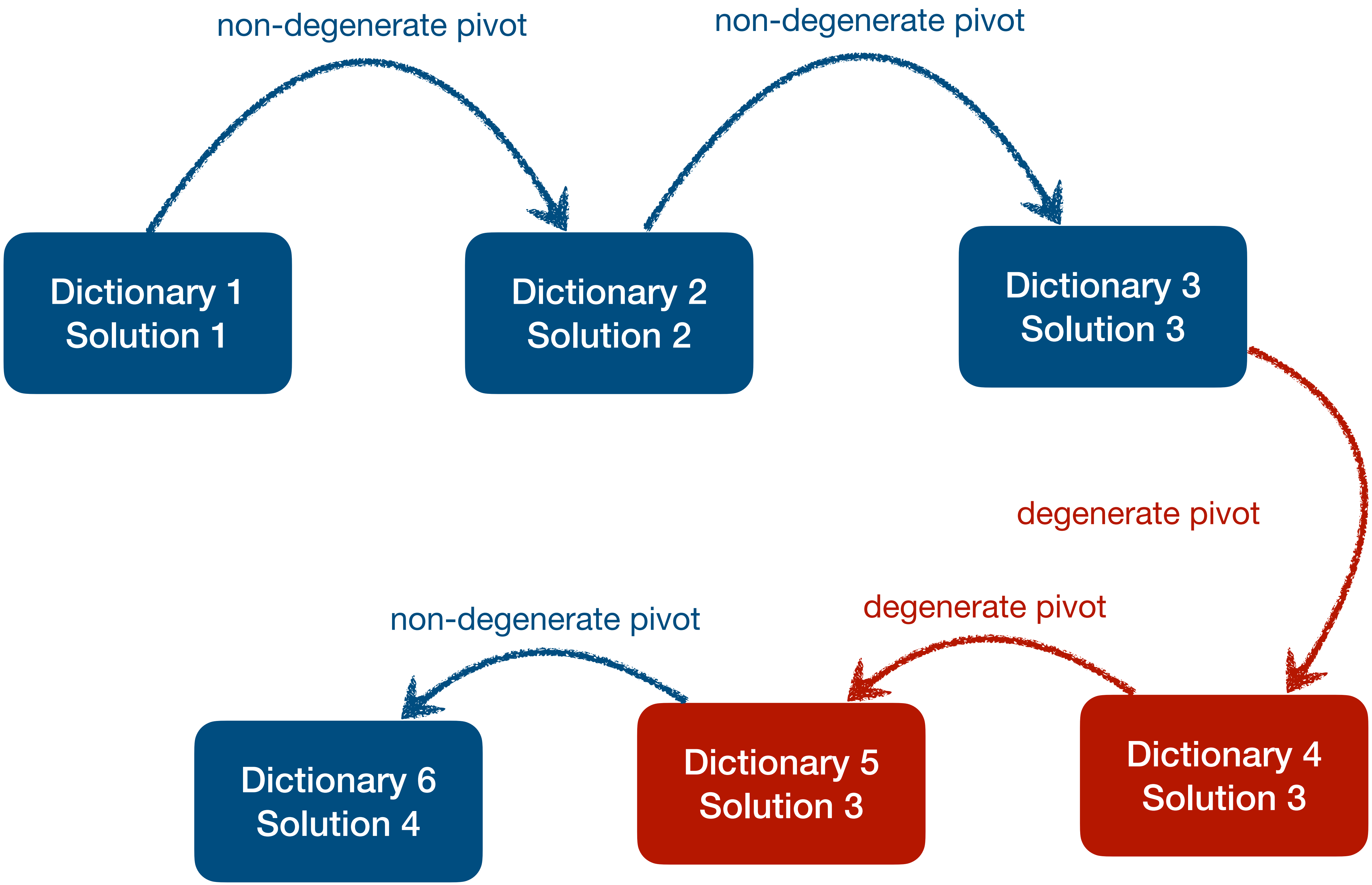

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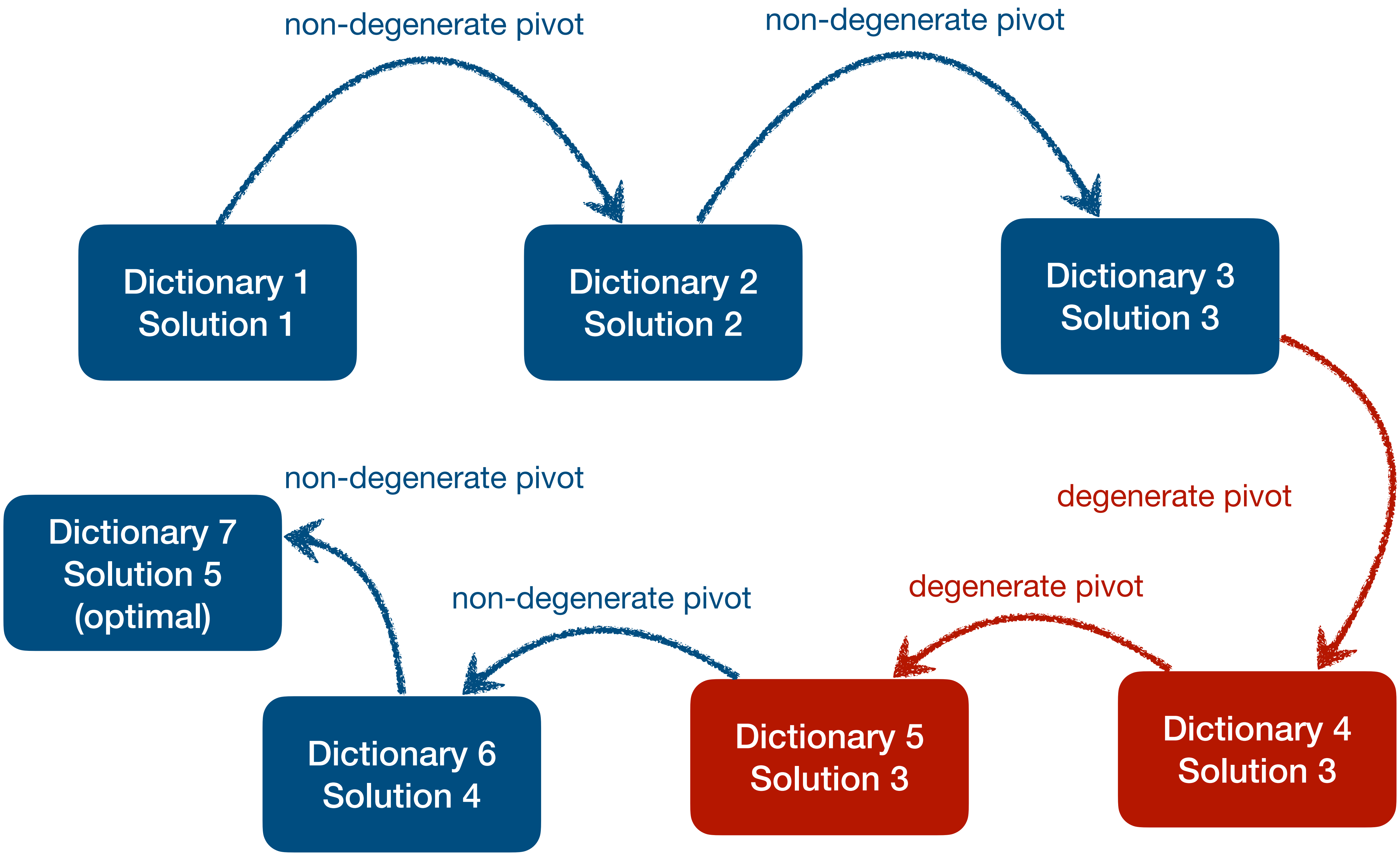

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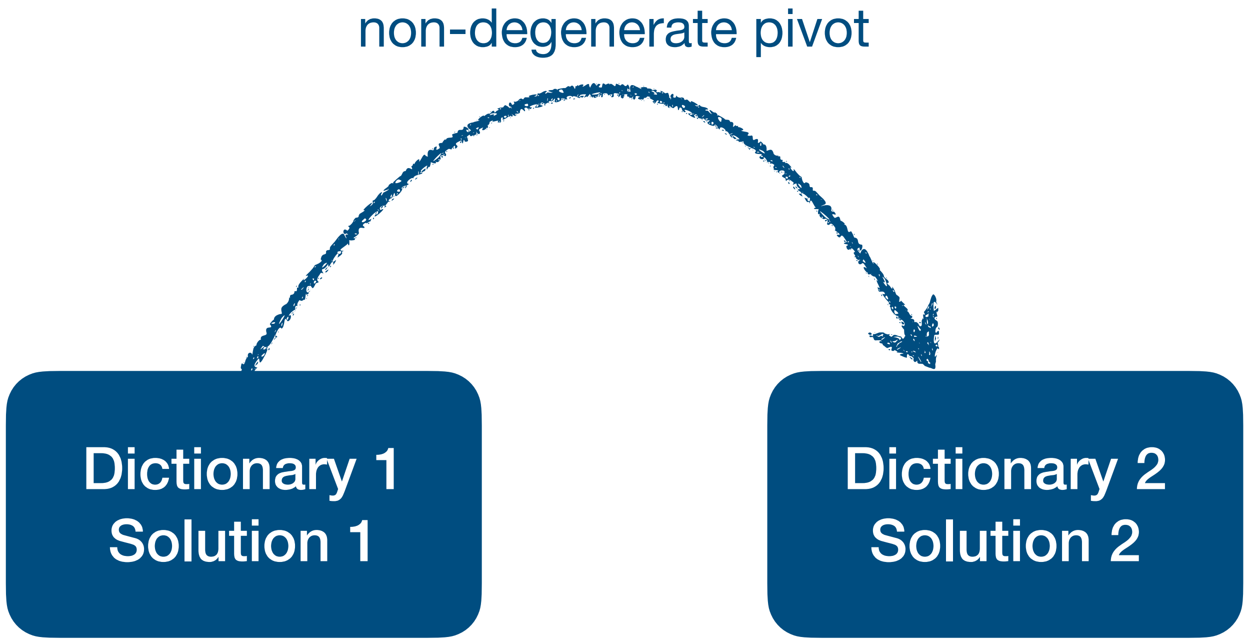

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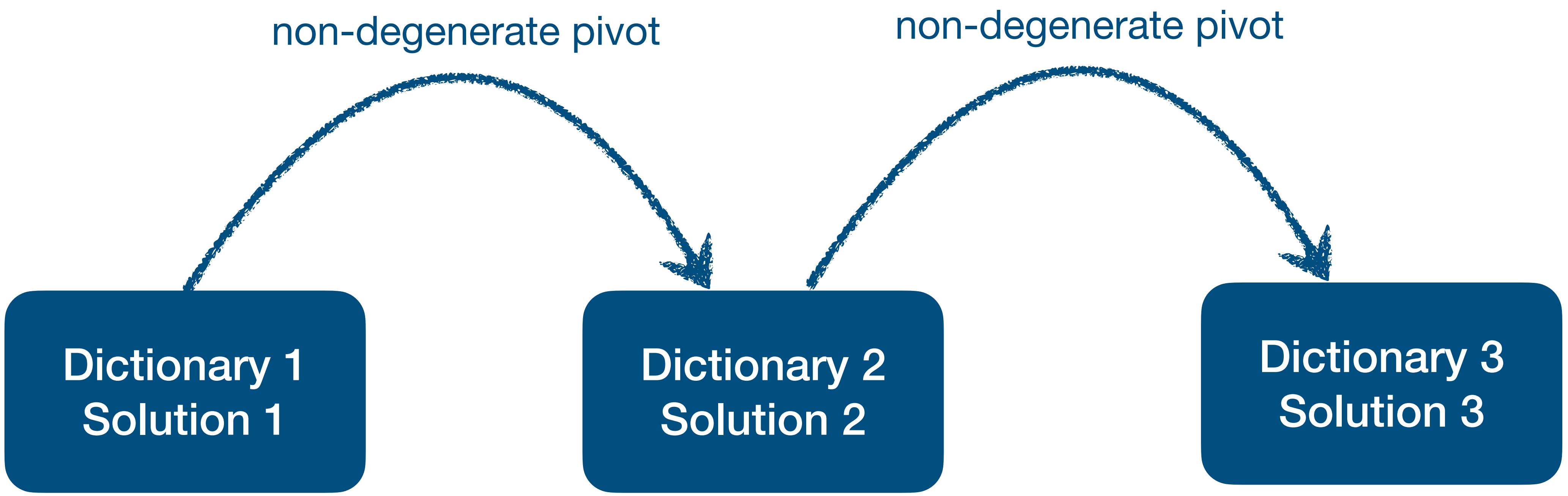

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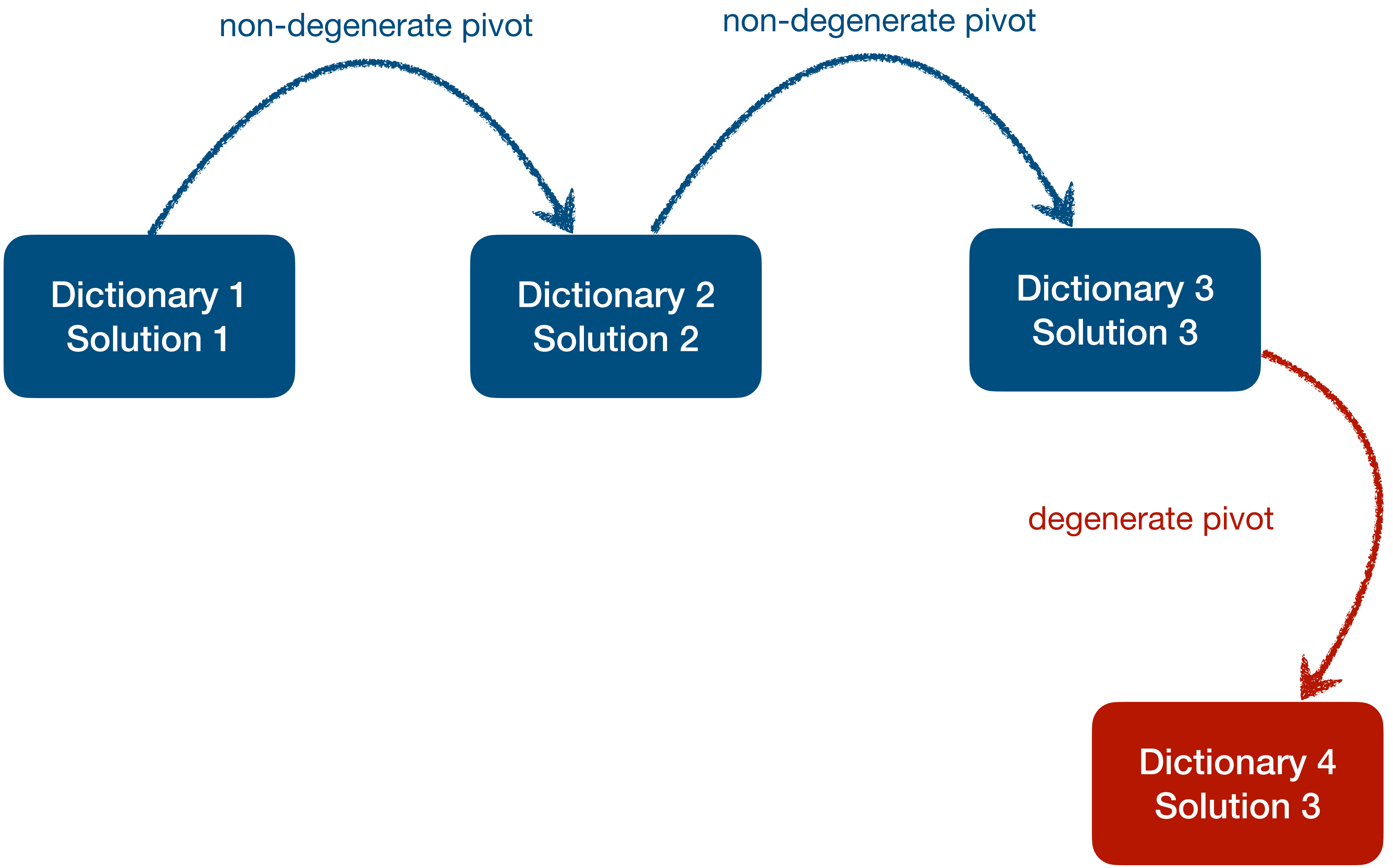

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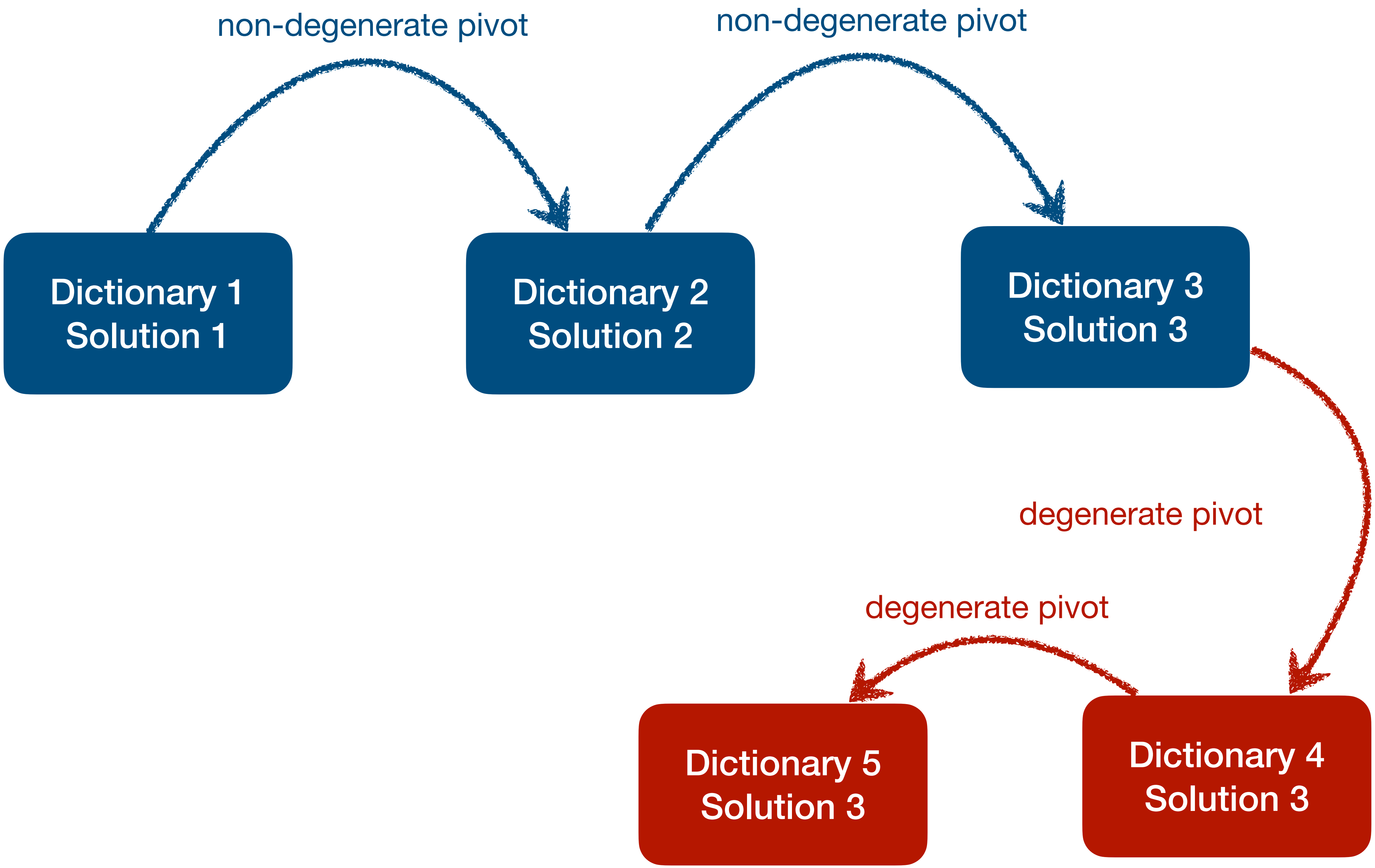

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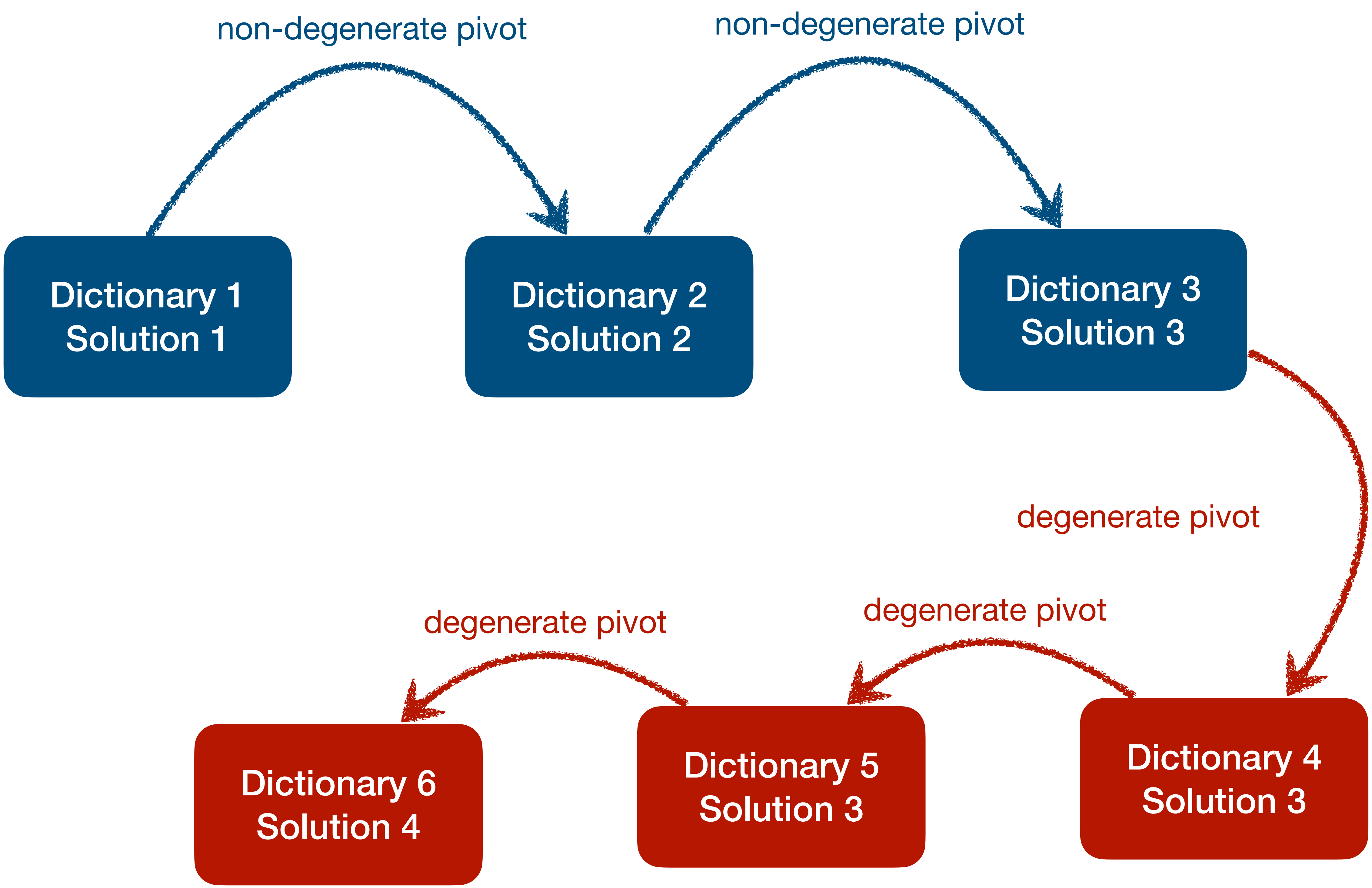

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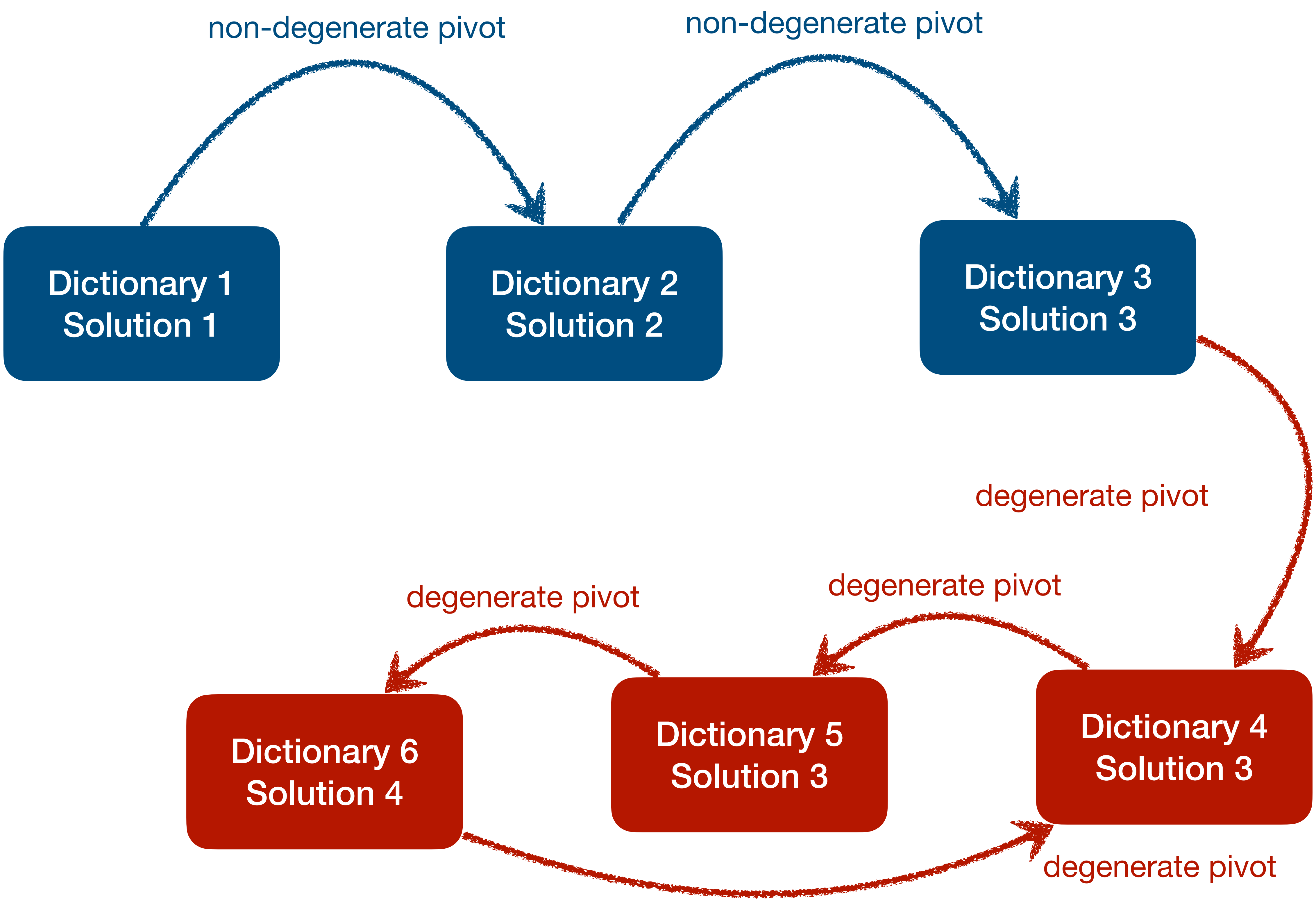

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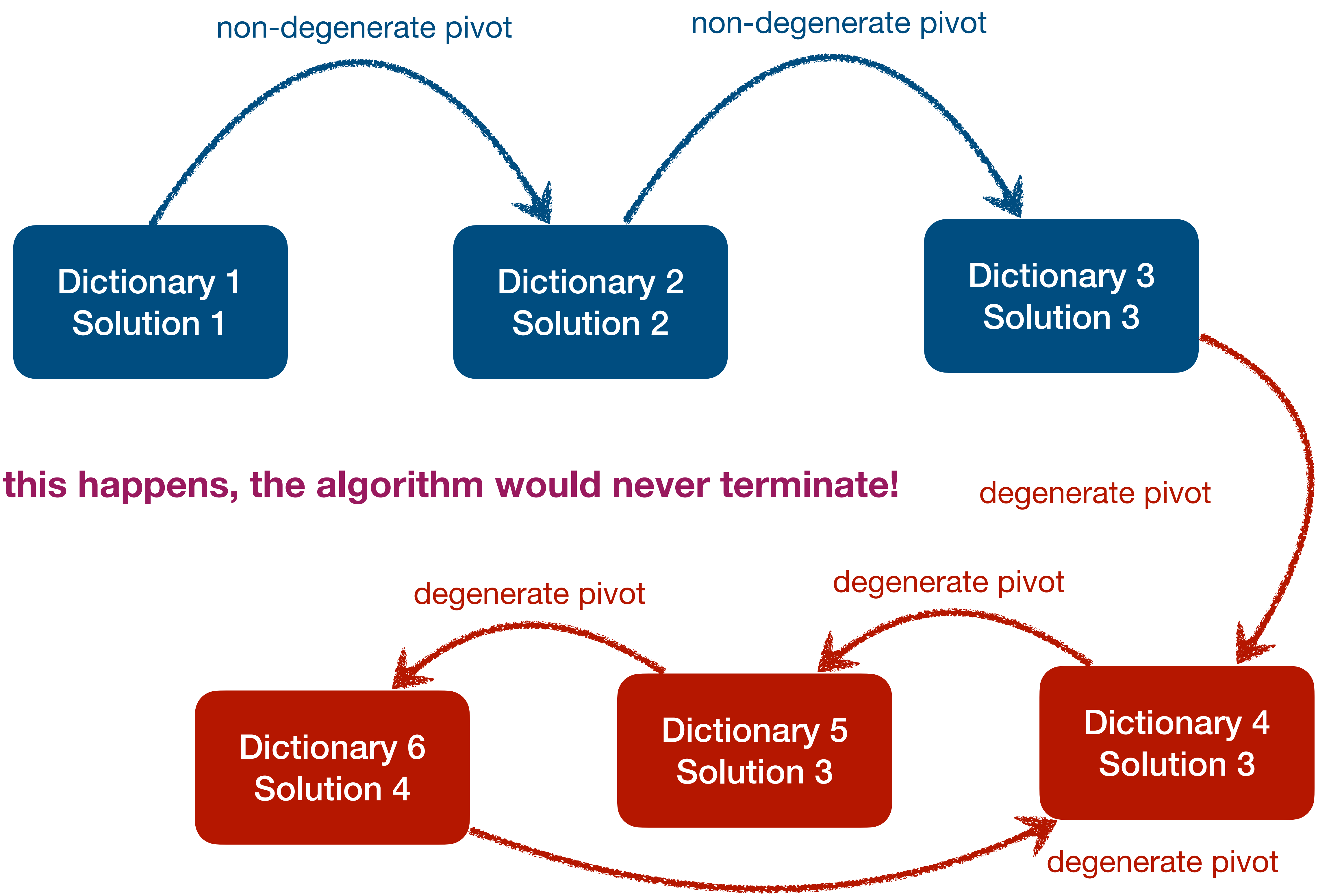

If this happens, the algorithm would never terminate!

Cycling

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In theory: Cycling can happen.

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In practice: Cycling rarely happens.

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But non-degenerate pivots are quite common.

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Can we avoid cycling in theory too?

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In practice: Cycling rarely happens.

But non-degenerate pivots are quite common.

Can we avoid cycling in theory too?

Bland's rule: For both the entering variable and the leaving variable, choose the one with the smallest index.

Termination

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Theorem: If the simplex method does not cycle, it terminates.

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Proof: A dictionary is determined by which variables are basic and which are non-basic.

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There only $\binom{n+m}{m} = \frac{(n+m)!}{n!m!}$ possibilities.

Geometry

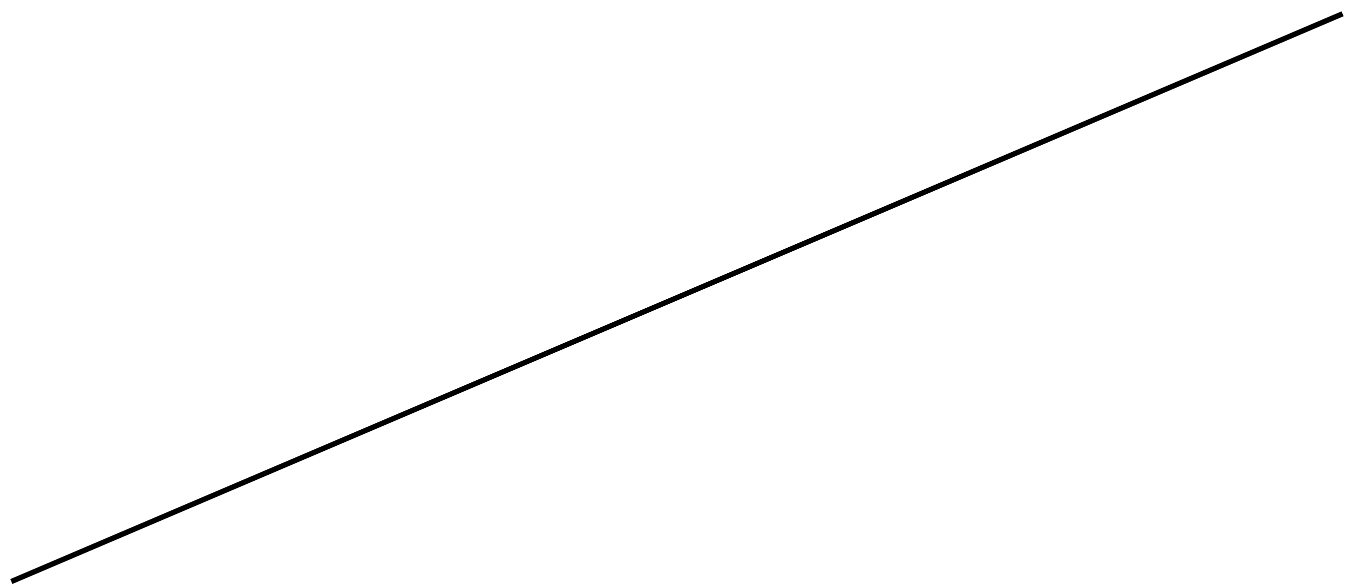
Every constraint corresponds to a hyperplane, which defines a halfspace.

The intersection of those halfspaces is the feasible region, which is a polyhedron (or polytope).

The part of each hyperplane that intersects with the feasible region is called a facet.

Geometry

Geometry



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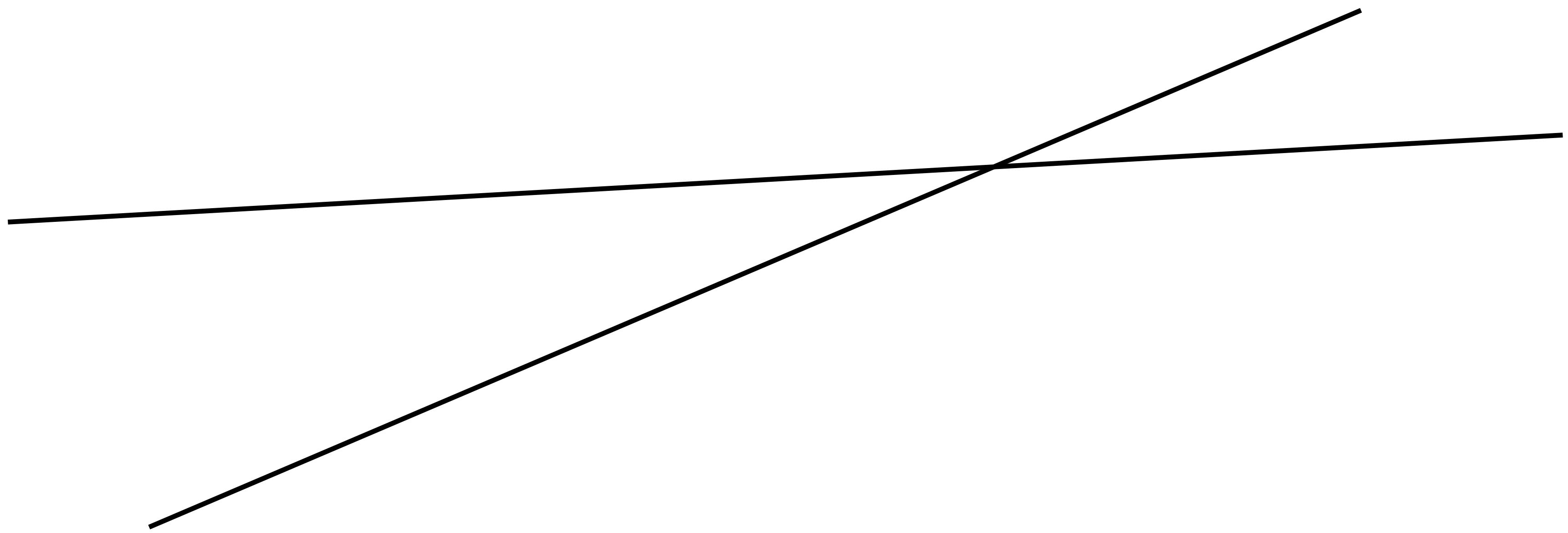

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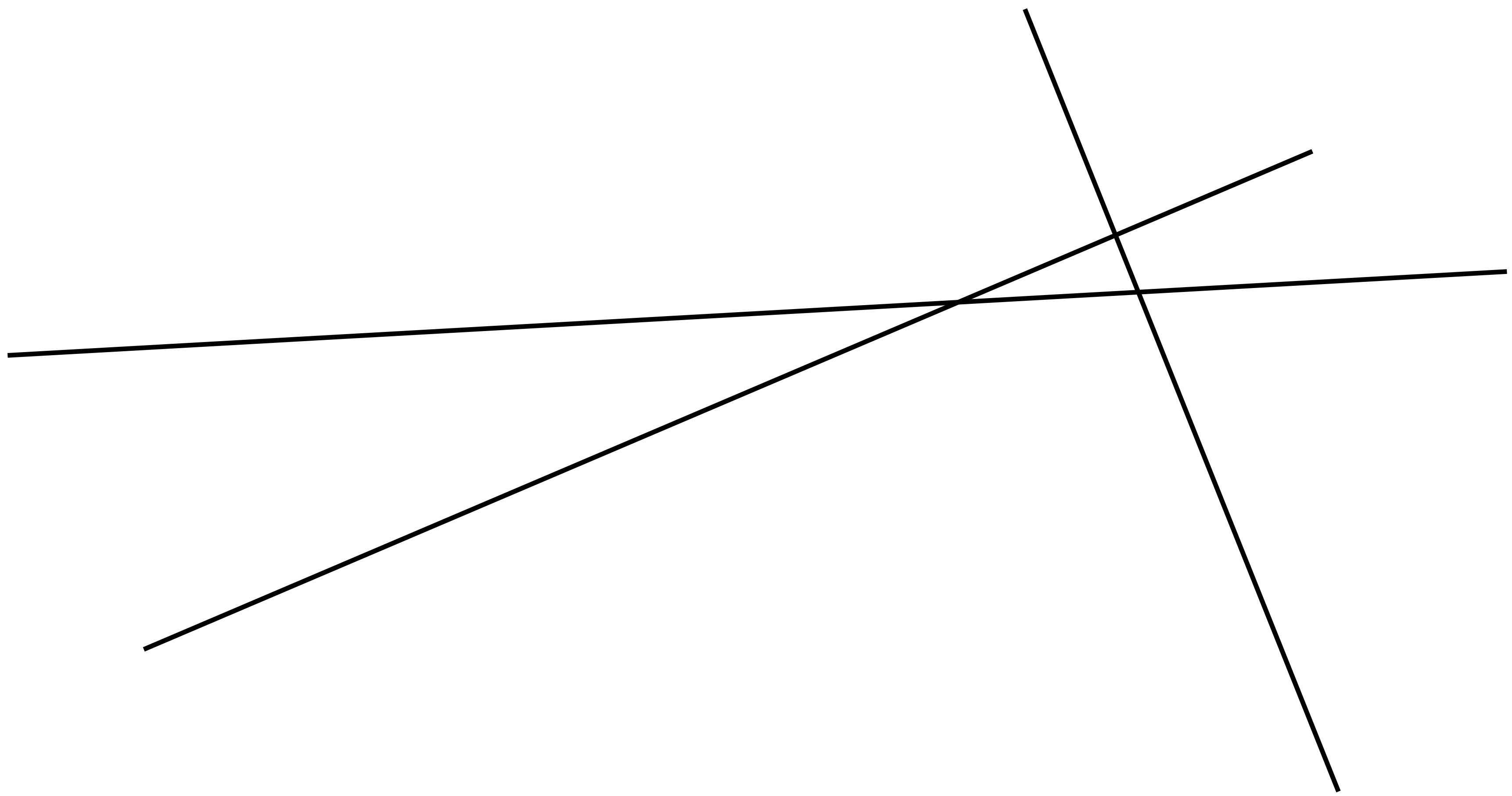
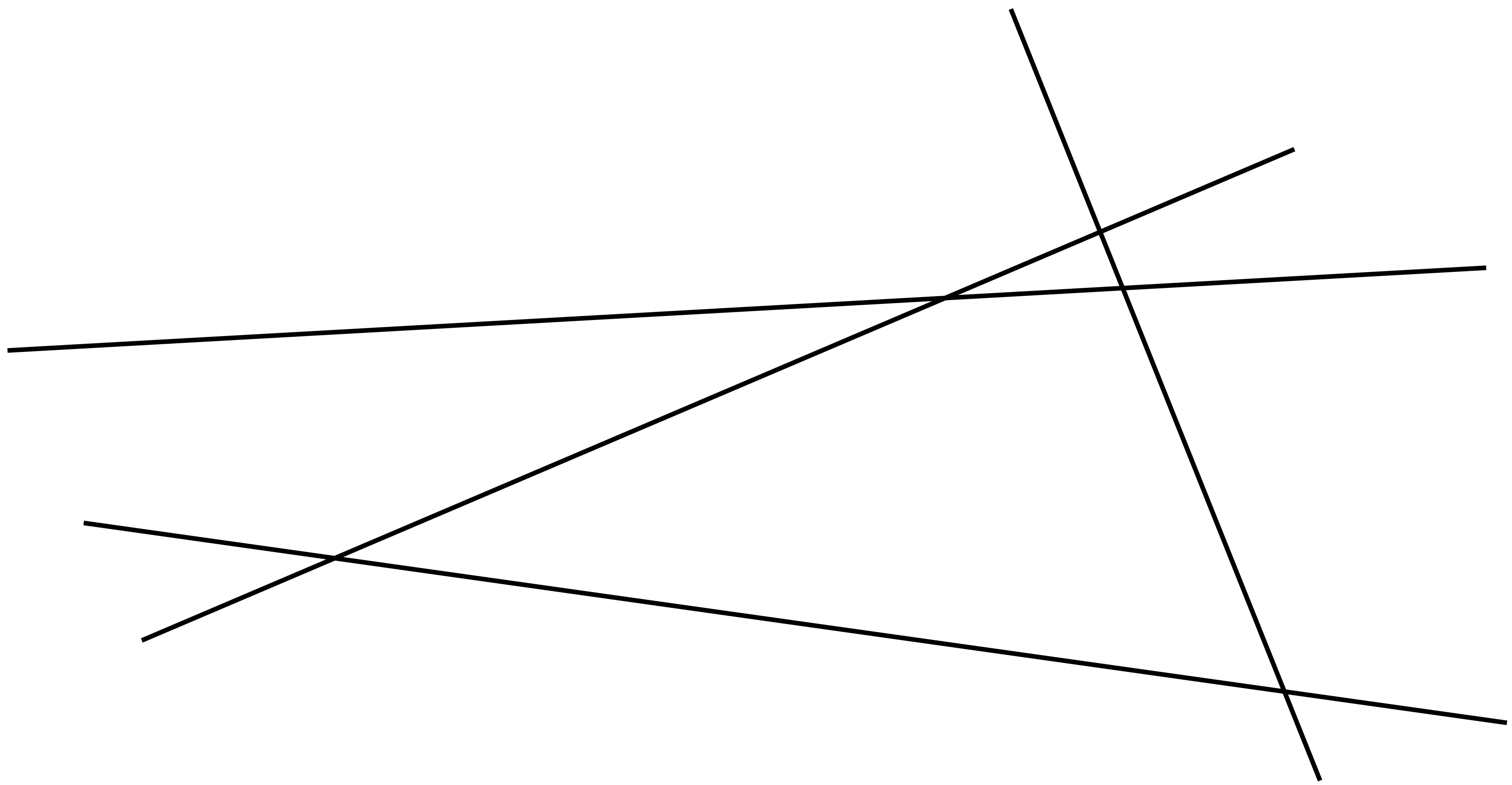


Geometry



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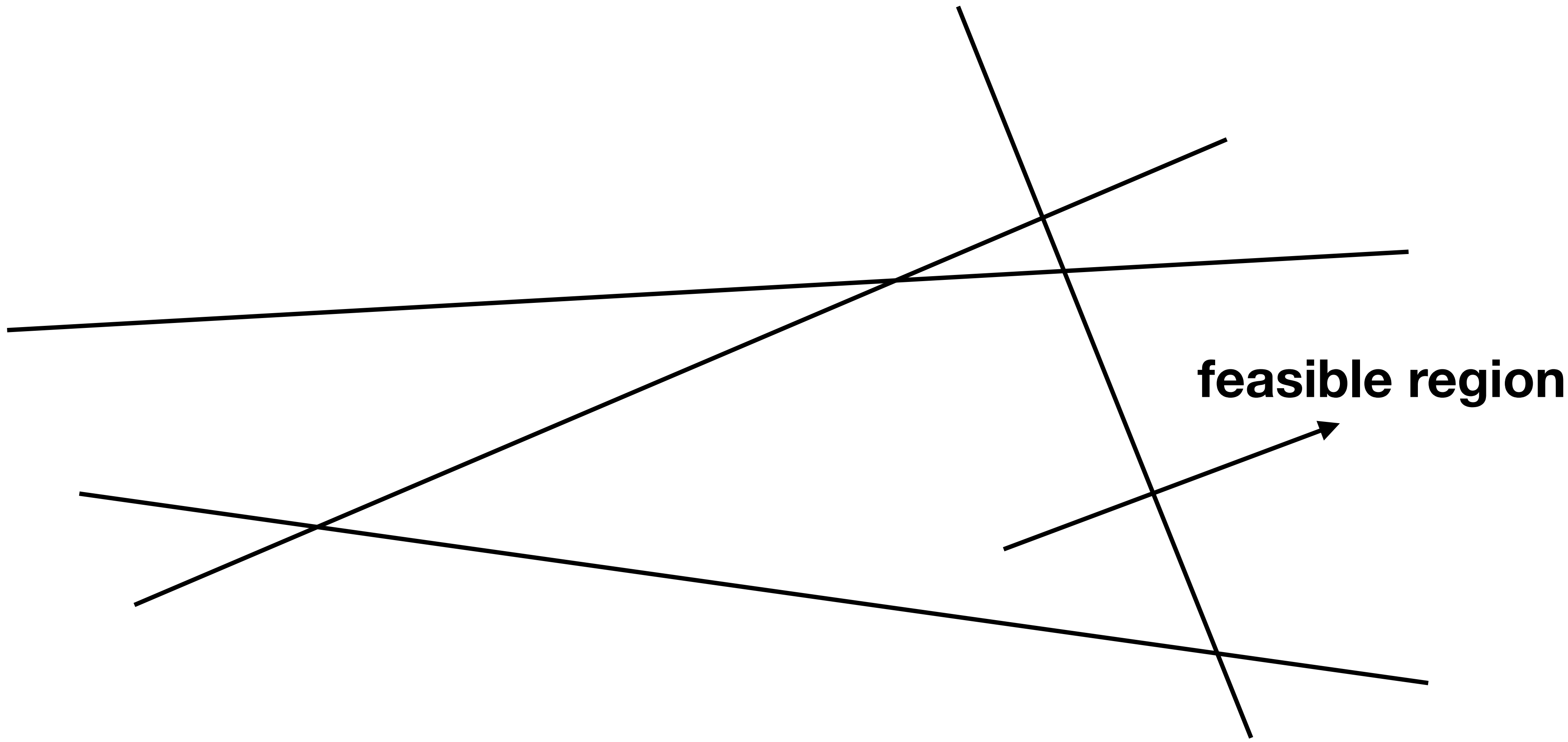

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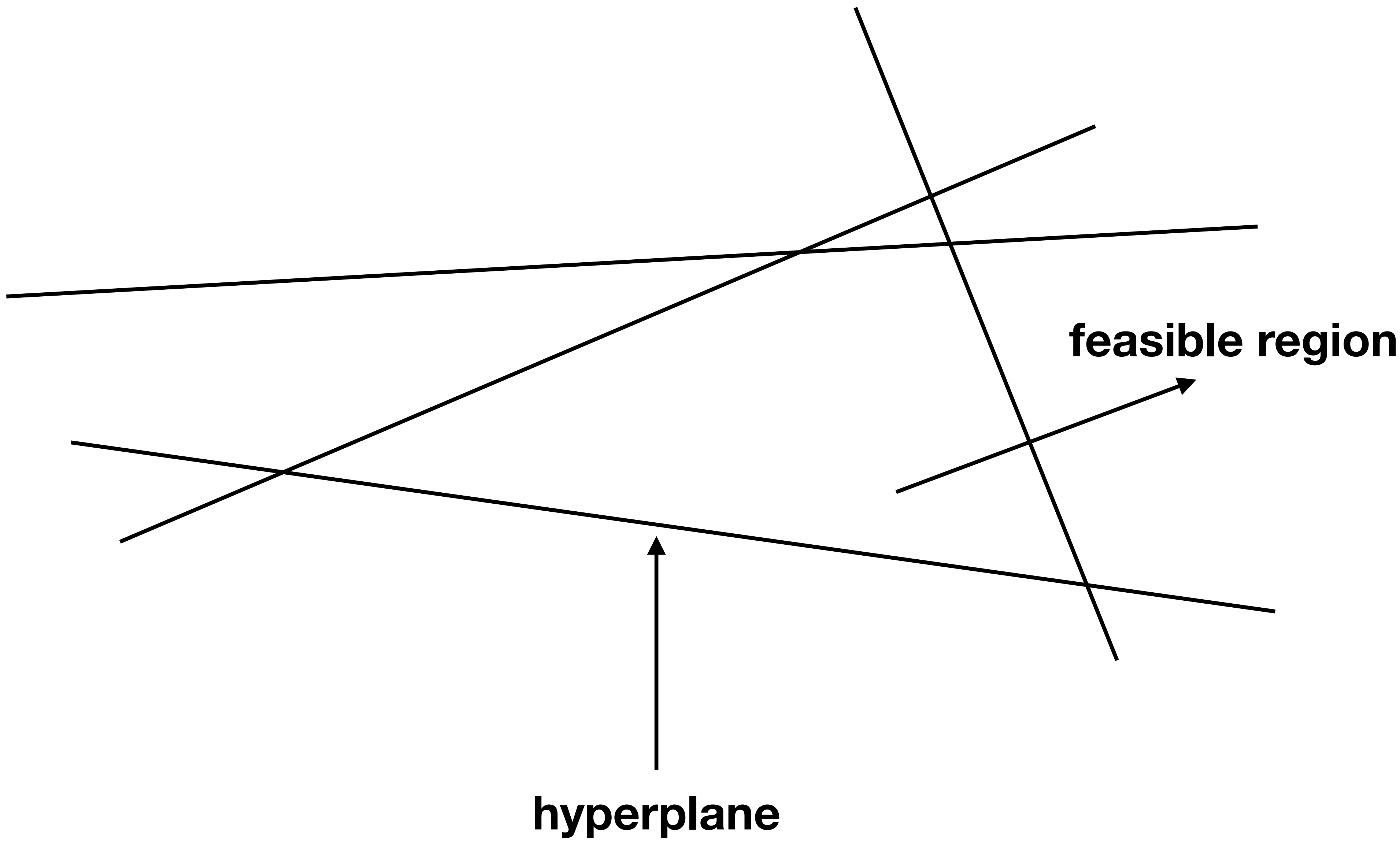

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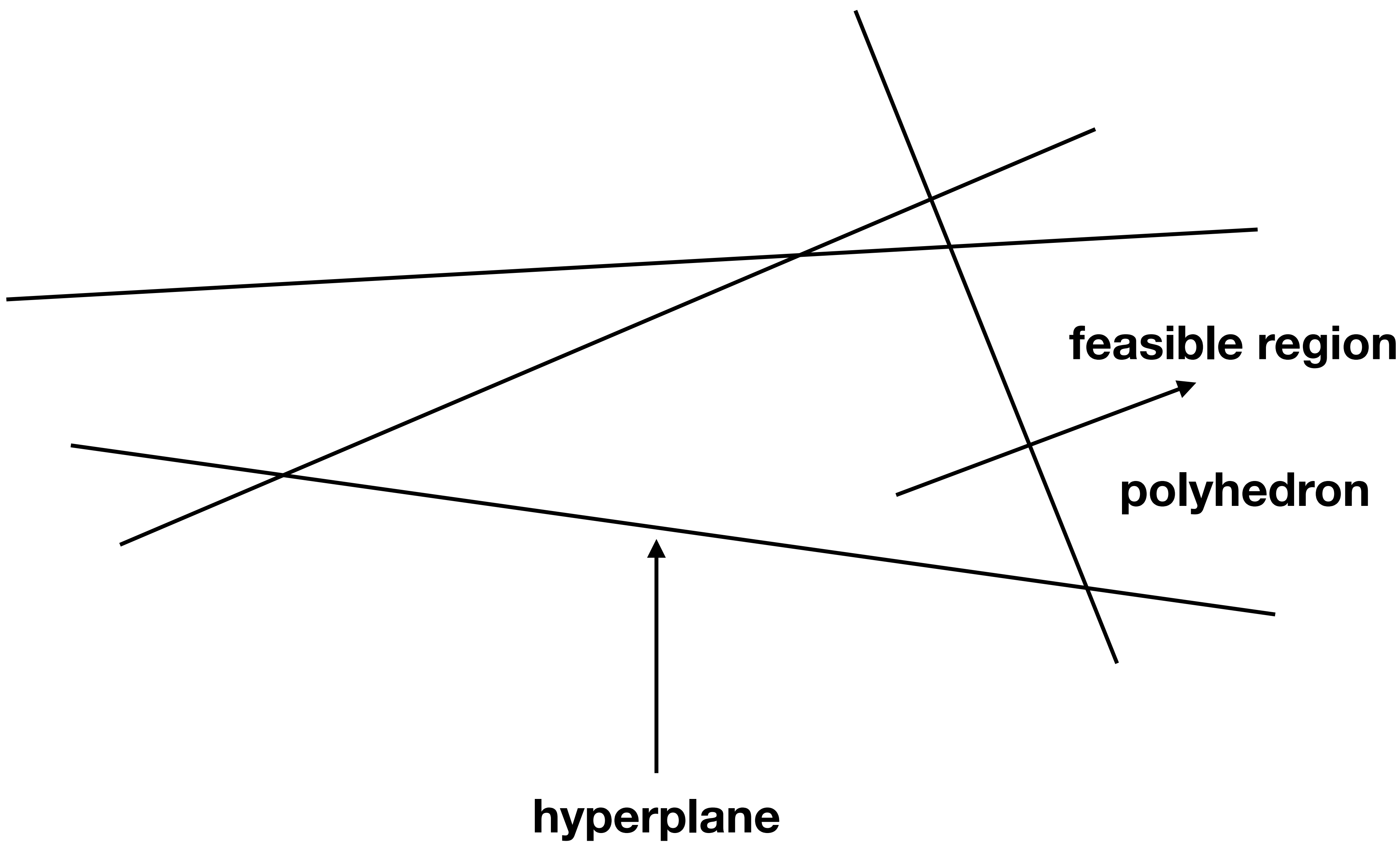

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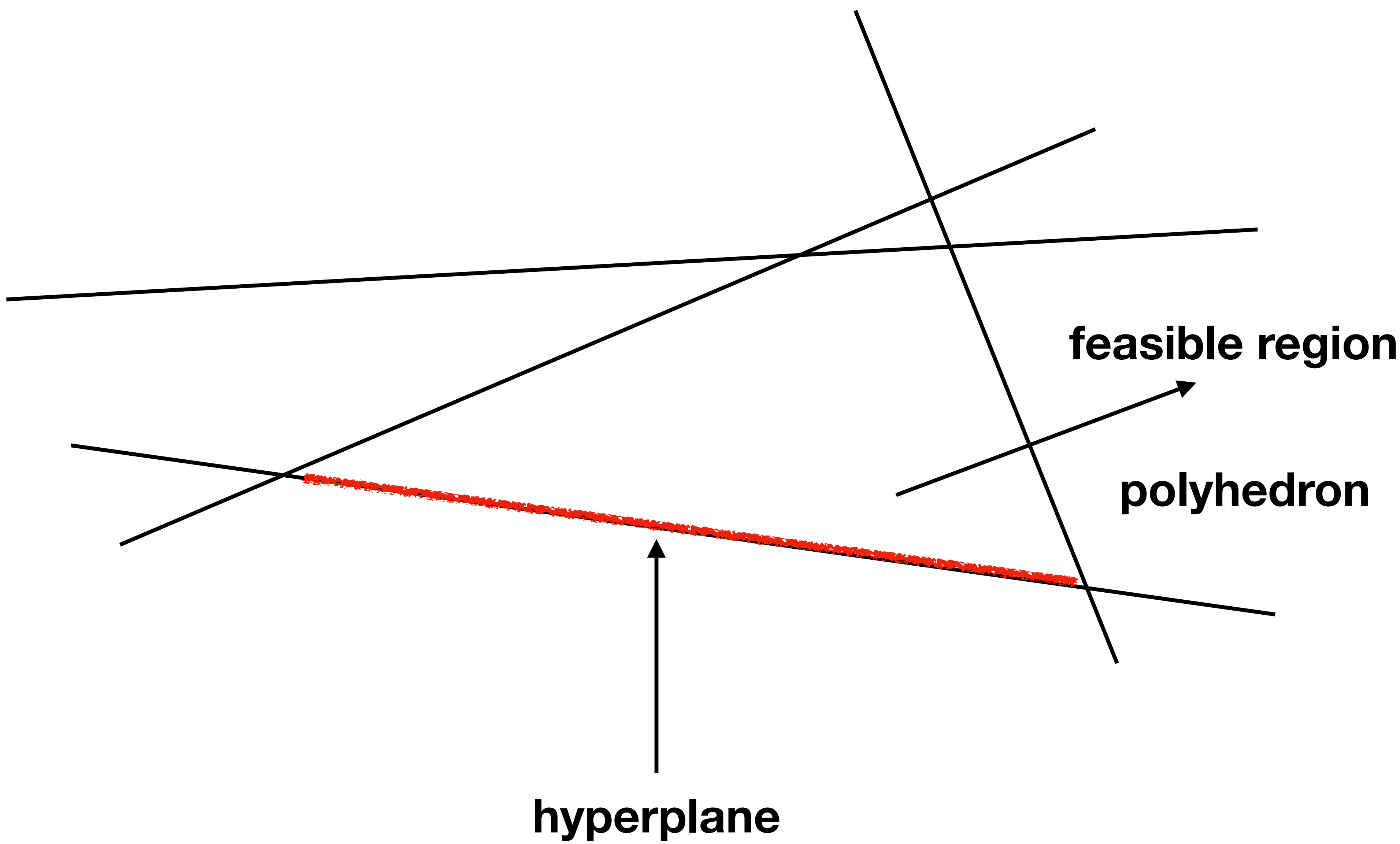

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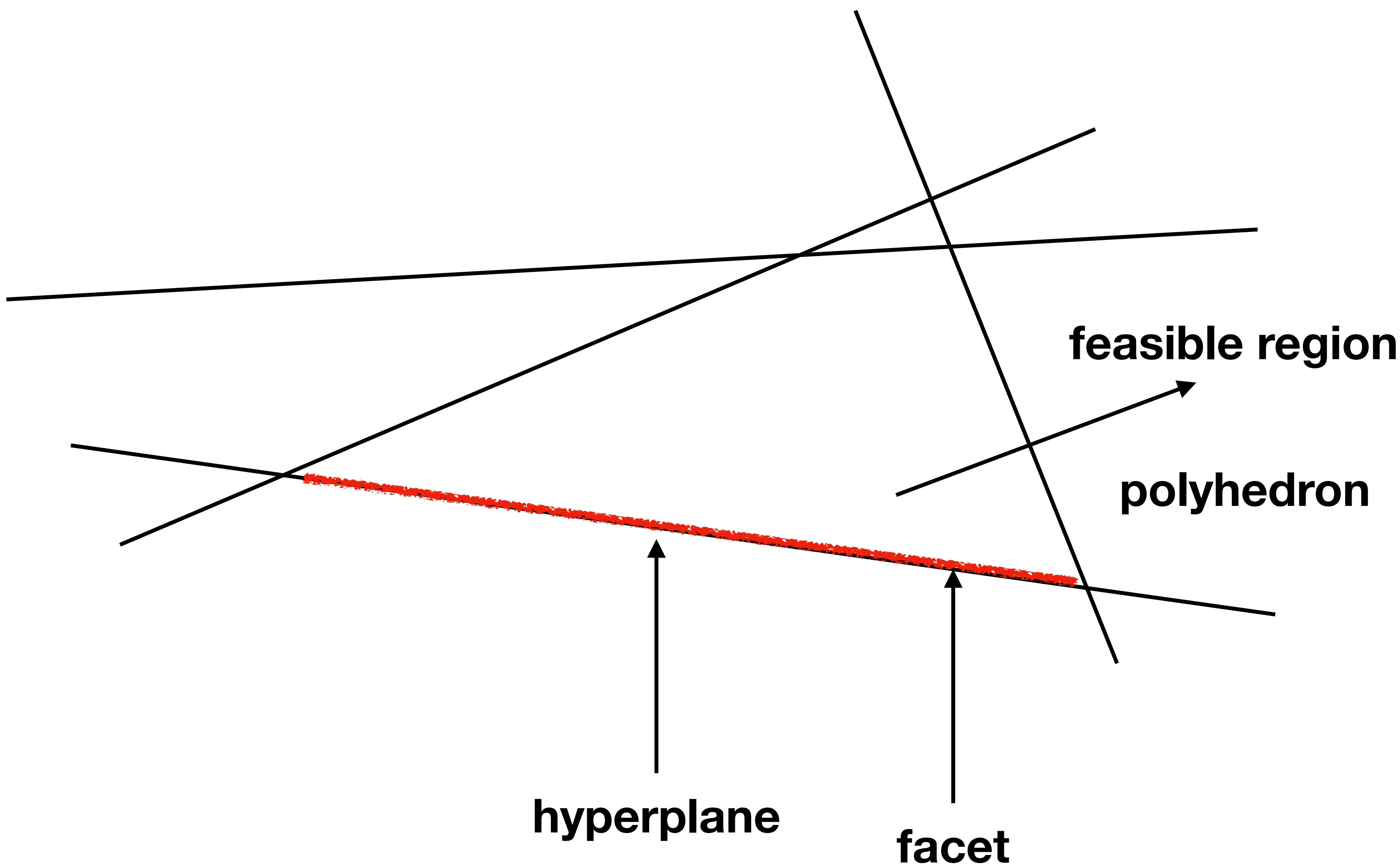

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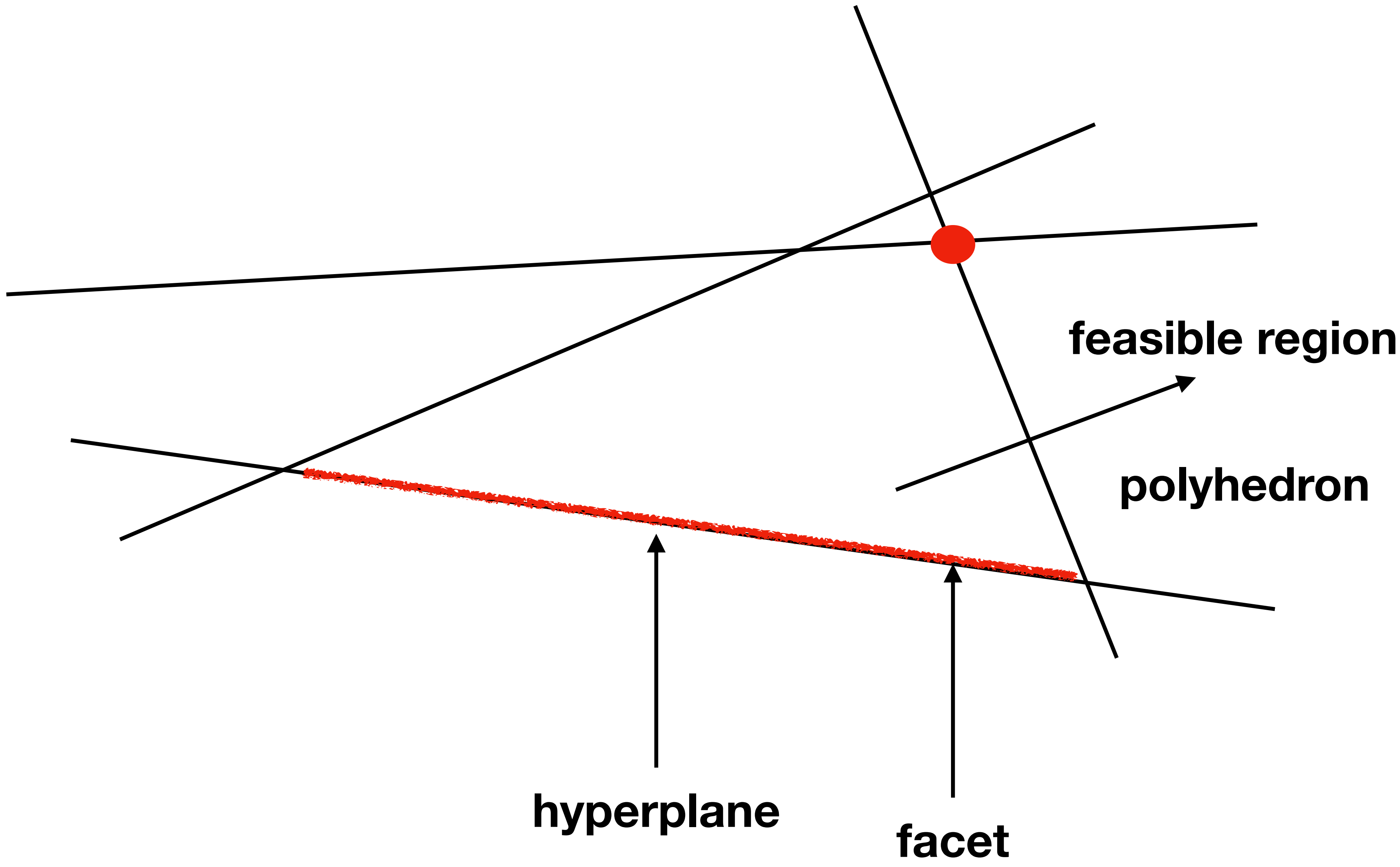

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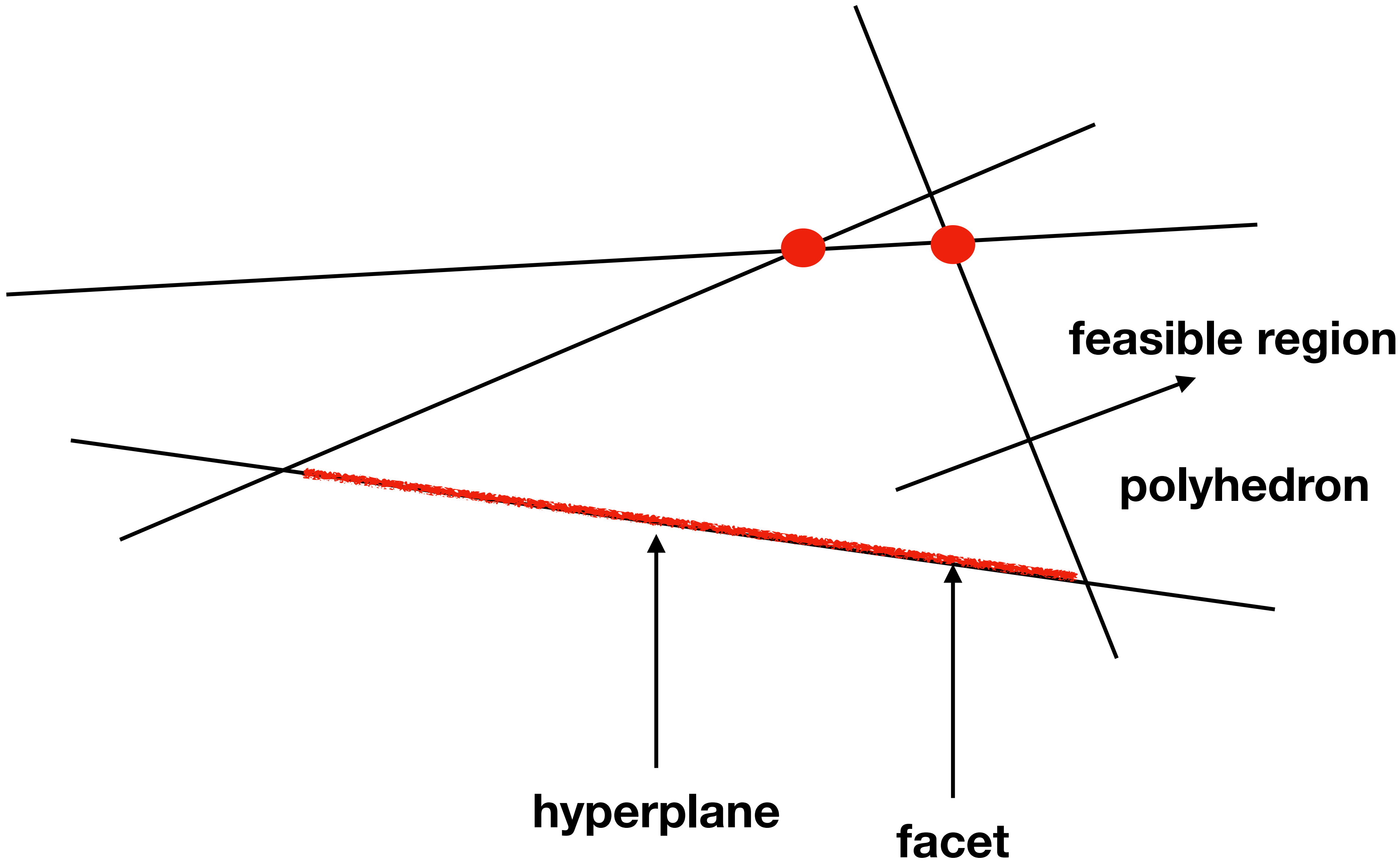

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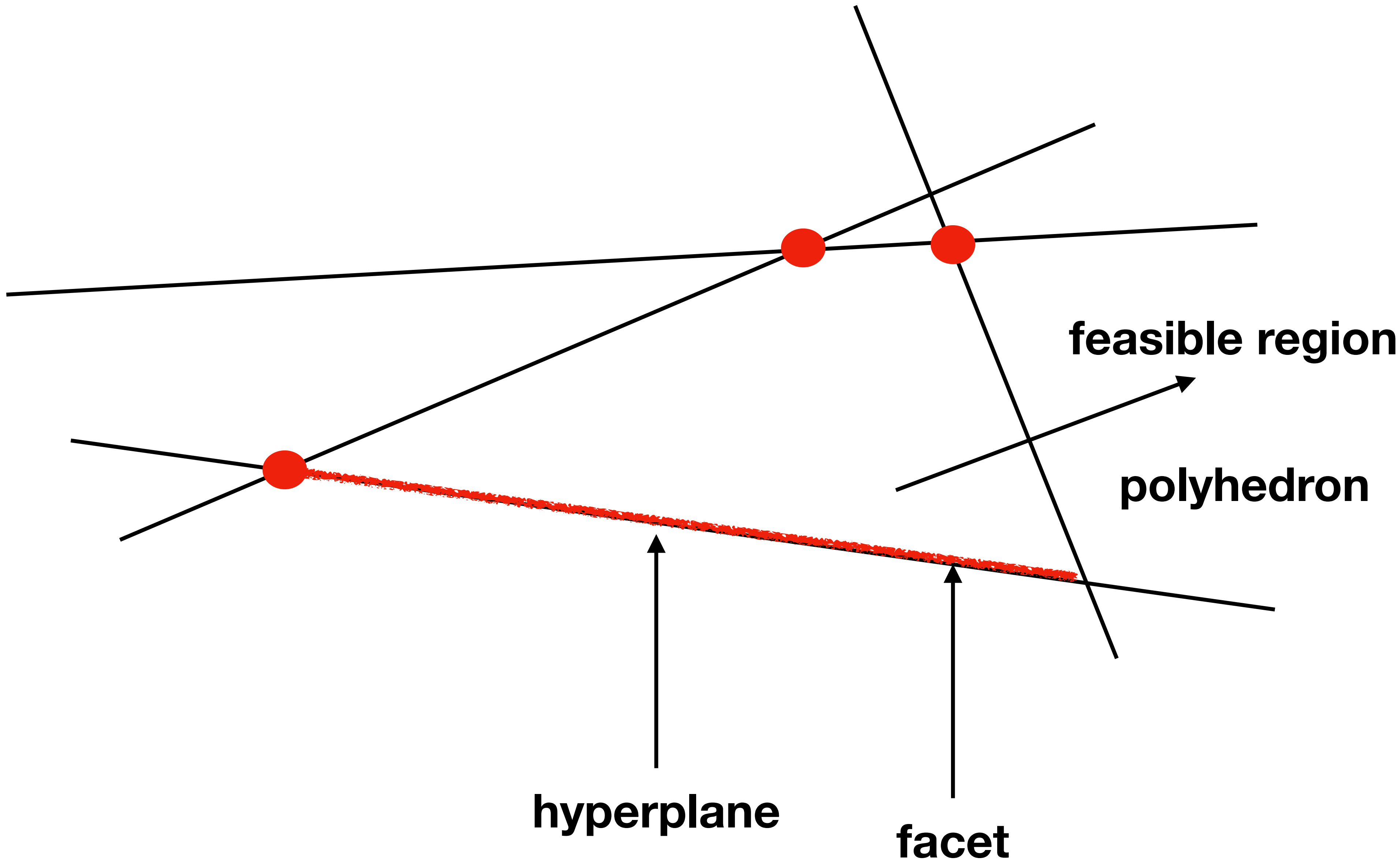

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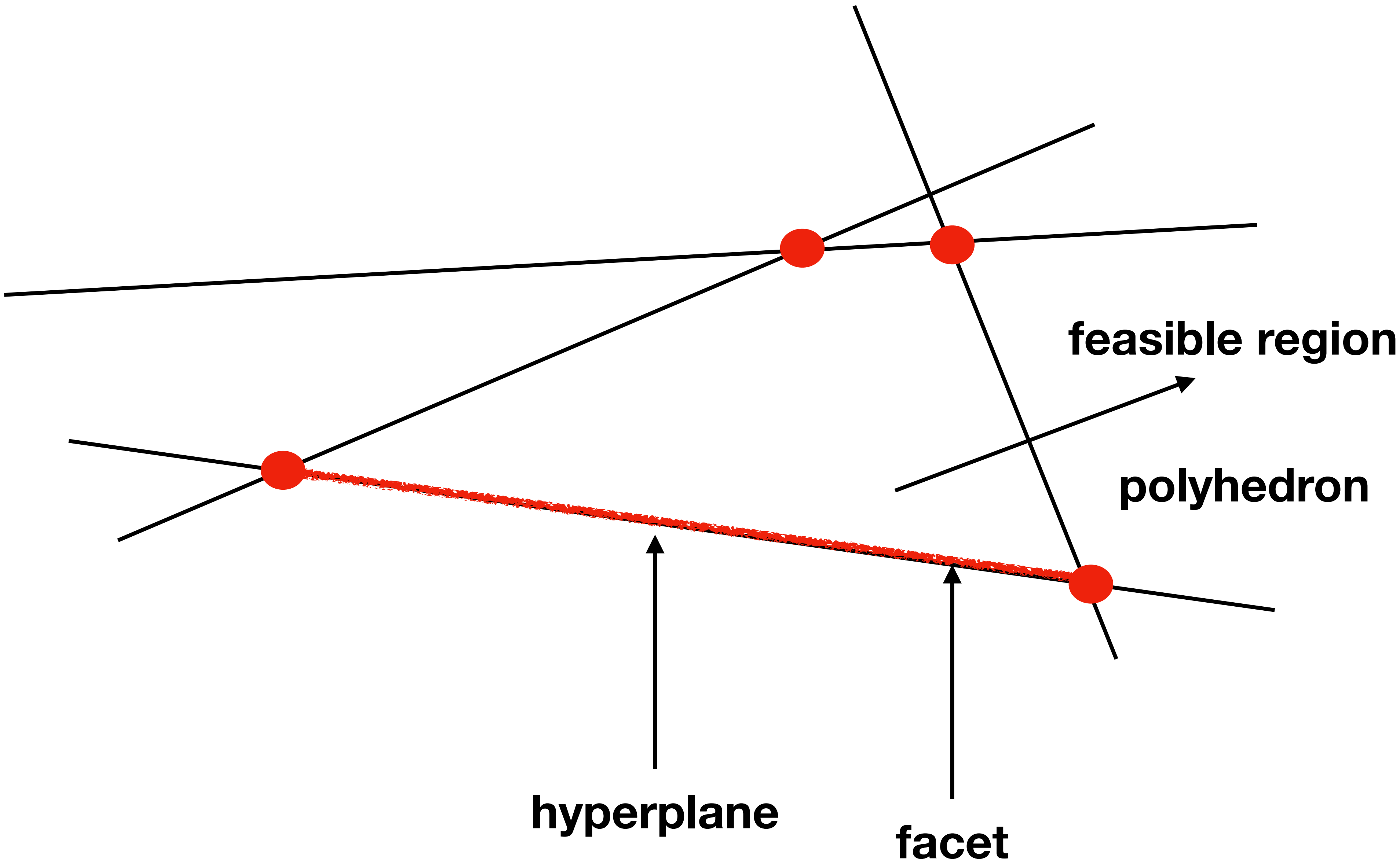

Geometry

Every constraint corresponds to a hyperplane, which defines a halfspace.

The intersection of those halfspaces is the feasible region, which is a polyhedron (or polytope).

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A facet corresponds to a constraint satisfied with equality, e.g., $x_1 + 2x_3 = 3$

In terms of the dictionary, a facet corresponds to the corresponding variable (original or slack) being 0.

Geometry

Consider an LP with three variables x_1, x_2, x_3.

In terms of the dictionary, a facet corresponds to the corresponding variable (original or slack) being 0.

An edge corresponds to two variables being 0.

A vertex corresponds to three variables being 0.

Geometry

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$\left.\begin{array}{l} 2x_1 + 3x_2 + x_3 \leq 5 \\ 4x_1 + x_2 + 2x + 3 \leq 11 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \end{array}\right\} \begin{array}{c} w_1 = w_2 = w_3 = 0 \\ \text{corresponds to the intersection of} \\ \text{these three hyperplanes.} \end{array}$$

$$x_1, x_2, x_3 \geq 0$$

Example

Maximise $x_1 + 2x_2 + 3x_3$

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$$x_1 + 2x_3 \leq 3$$
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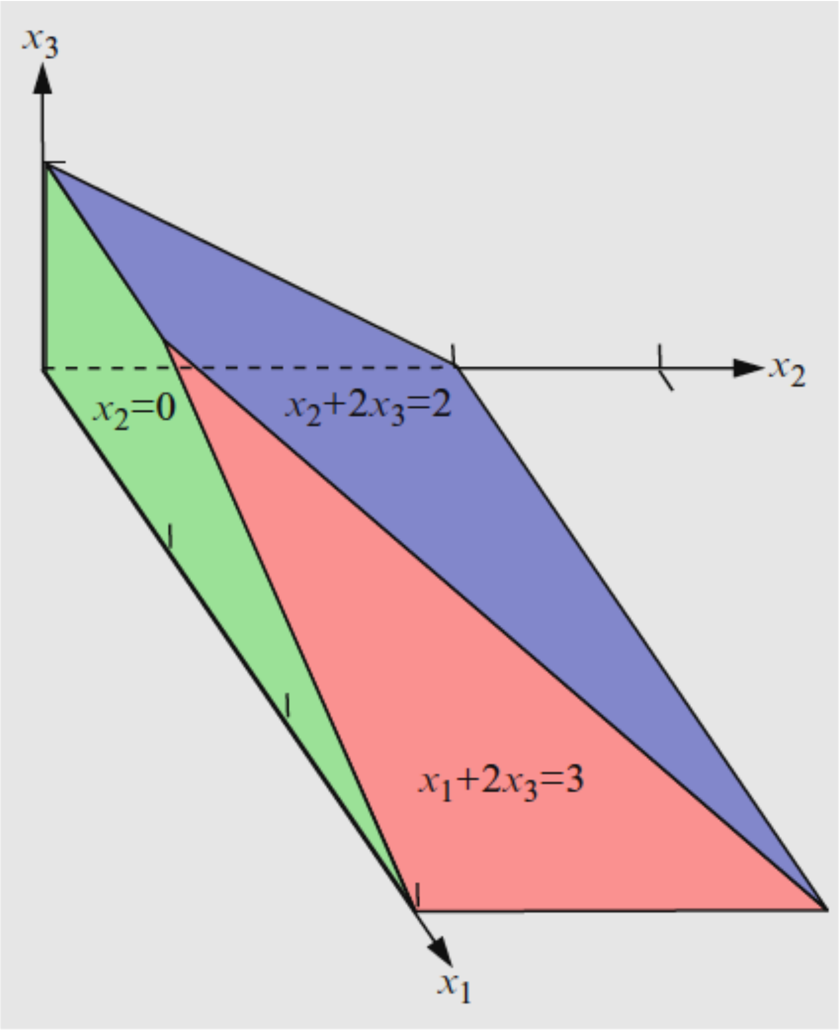

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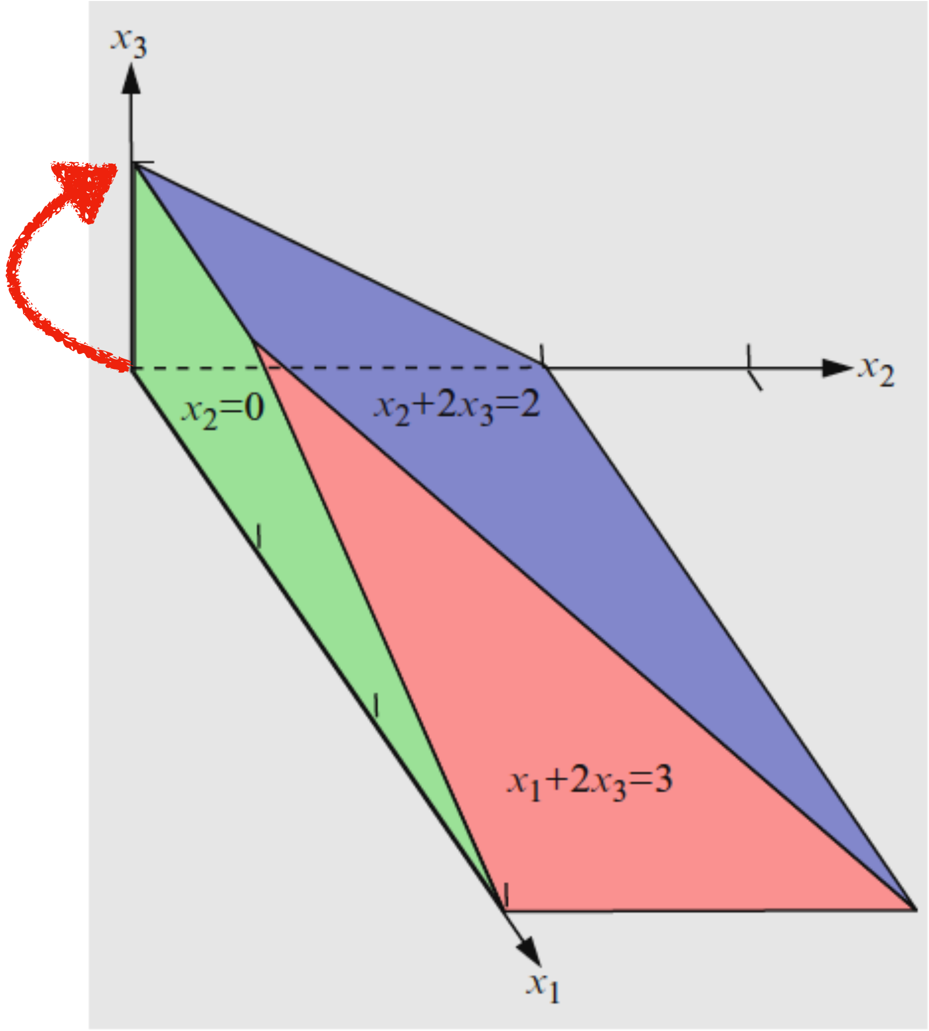

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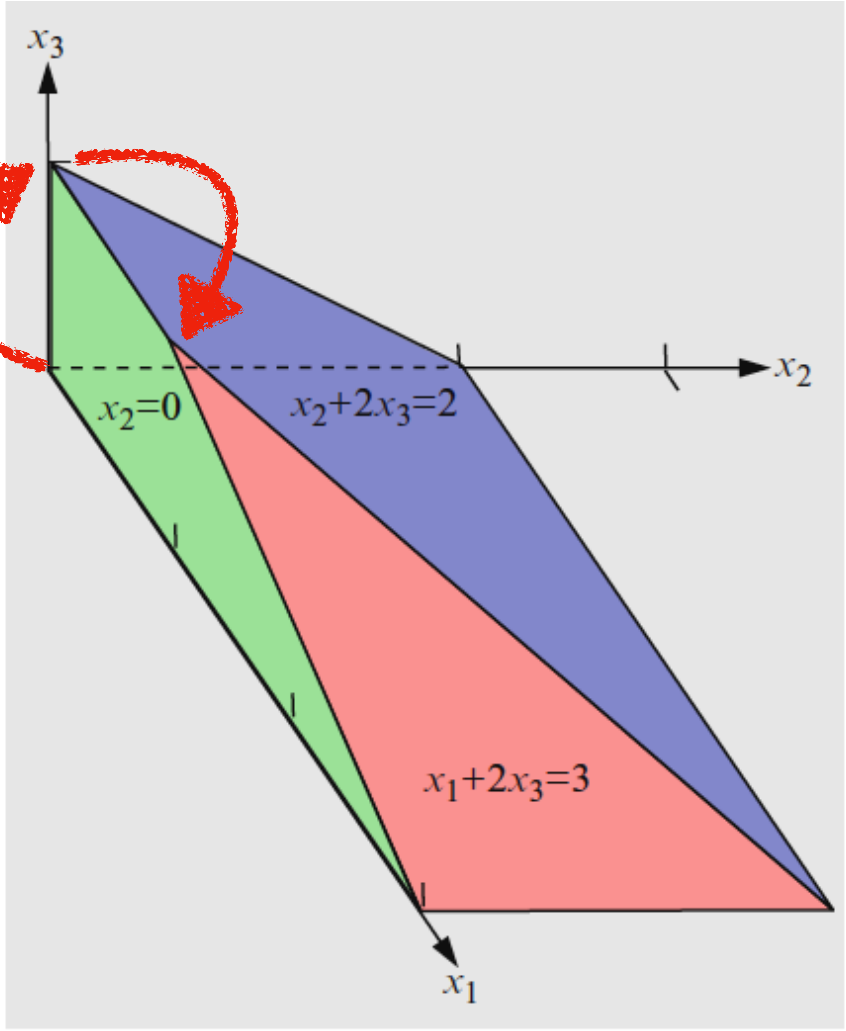

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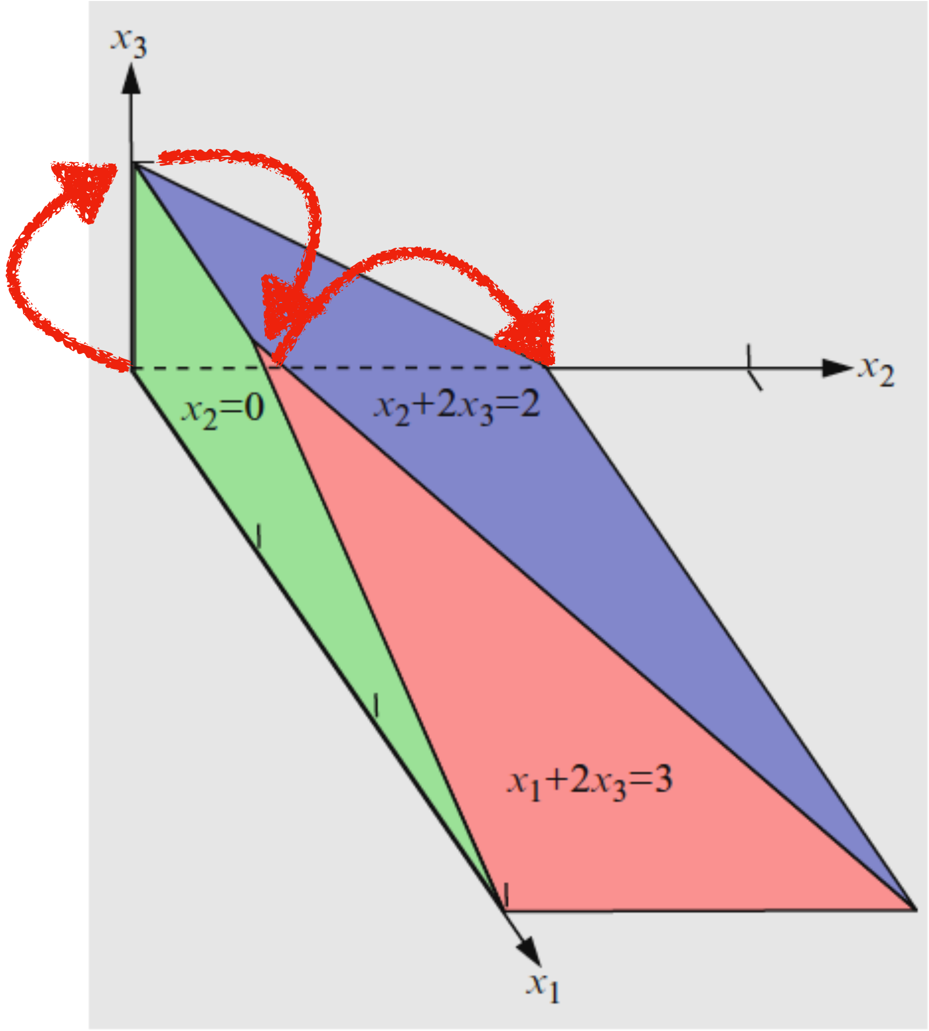

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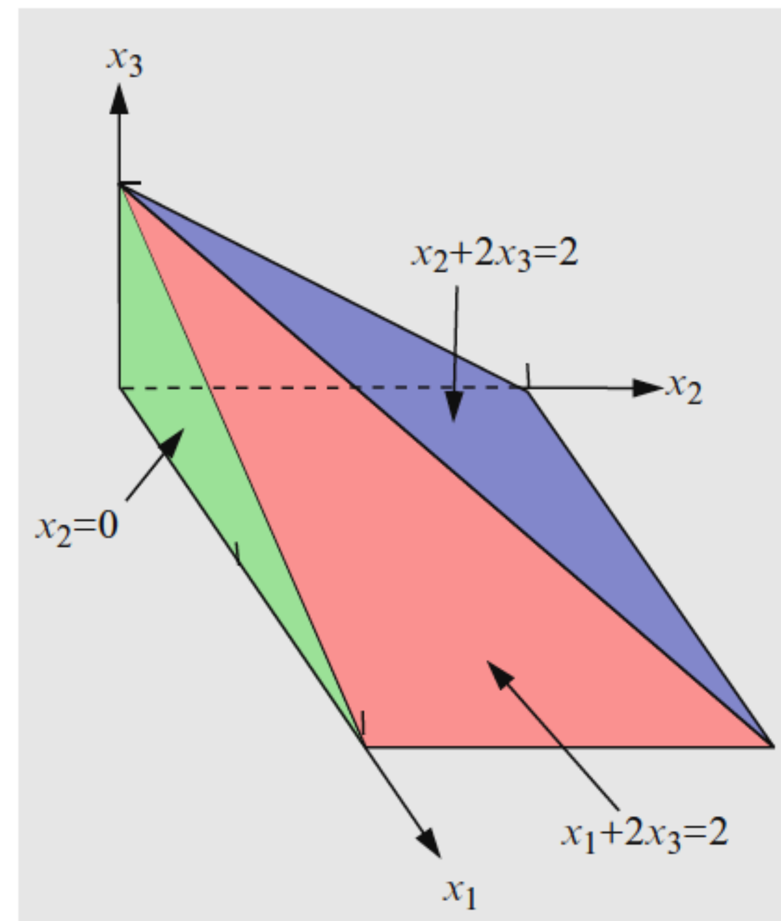
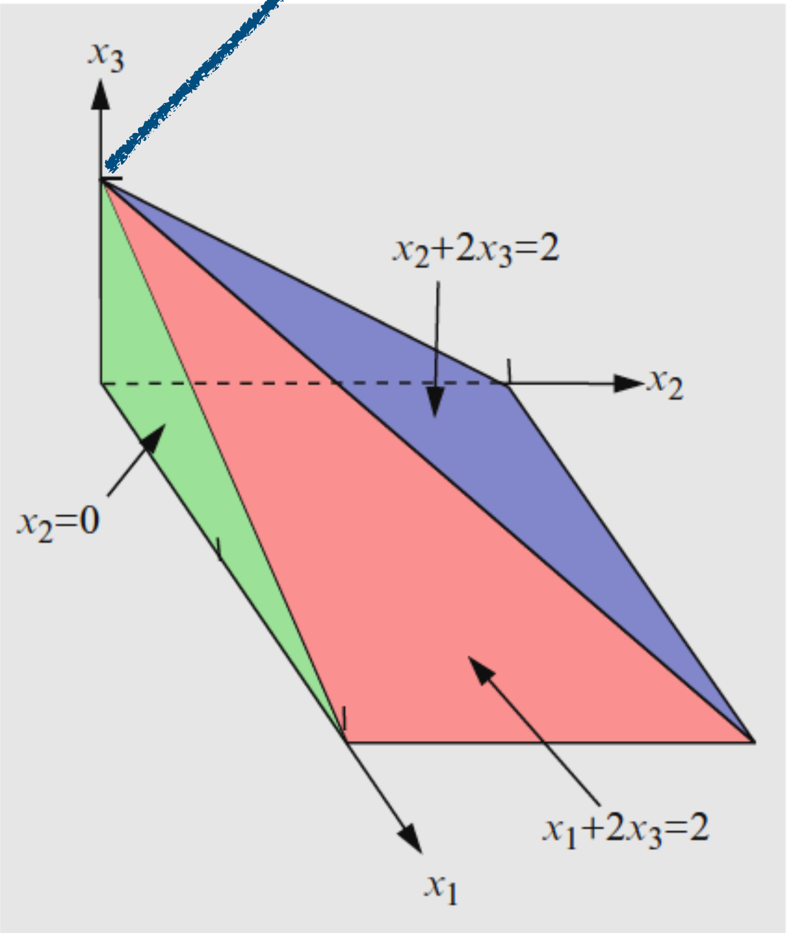


Example

intersection of four facets



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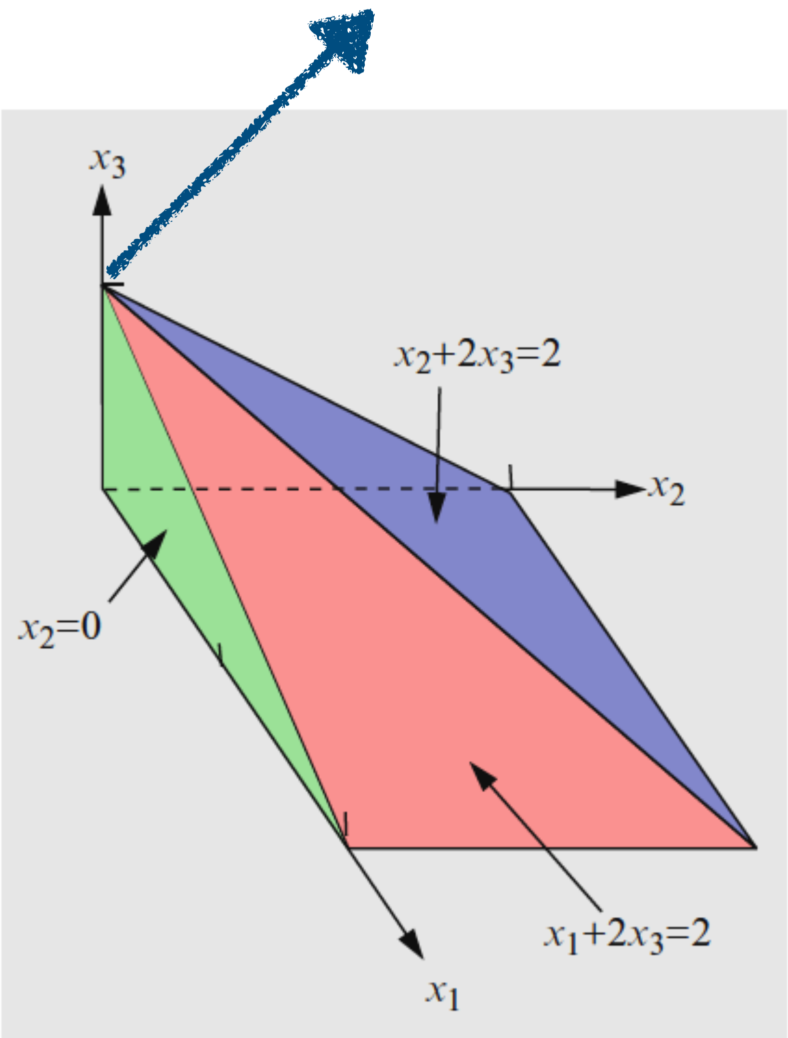

The intersection point corresponds to the same solution of the LP.

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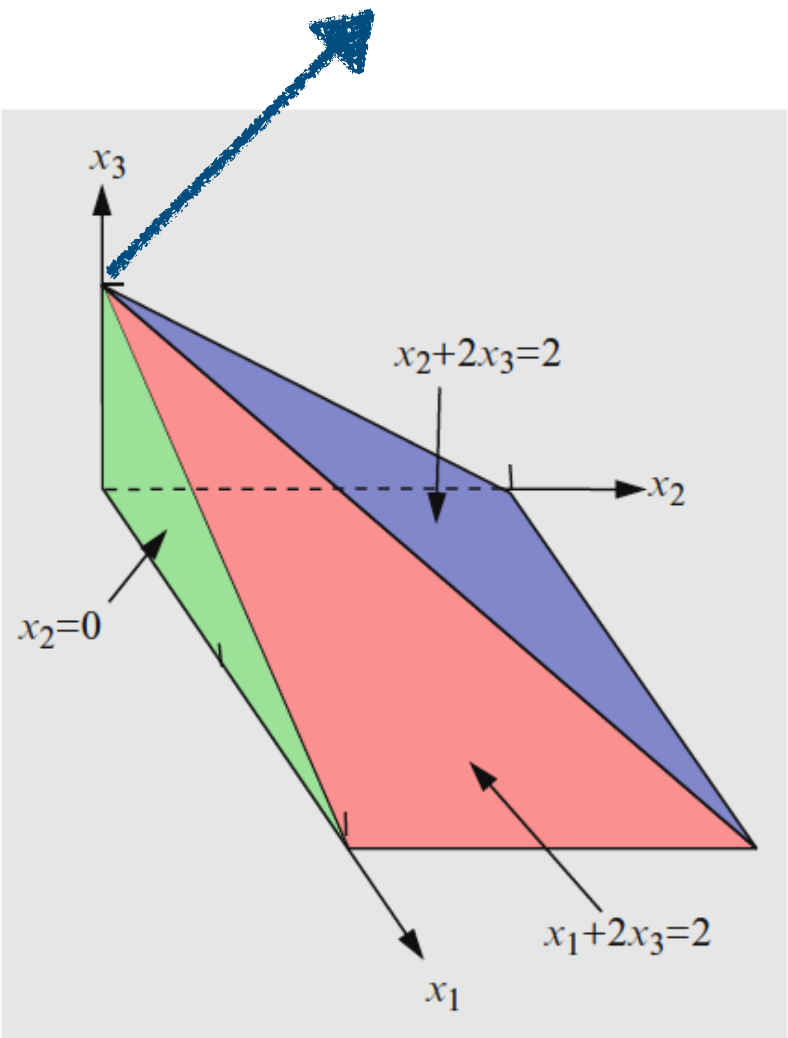

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But it corresponds to four different basic feasible solutions/dictionaries.

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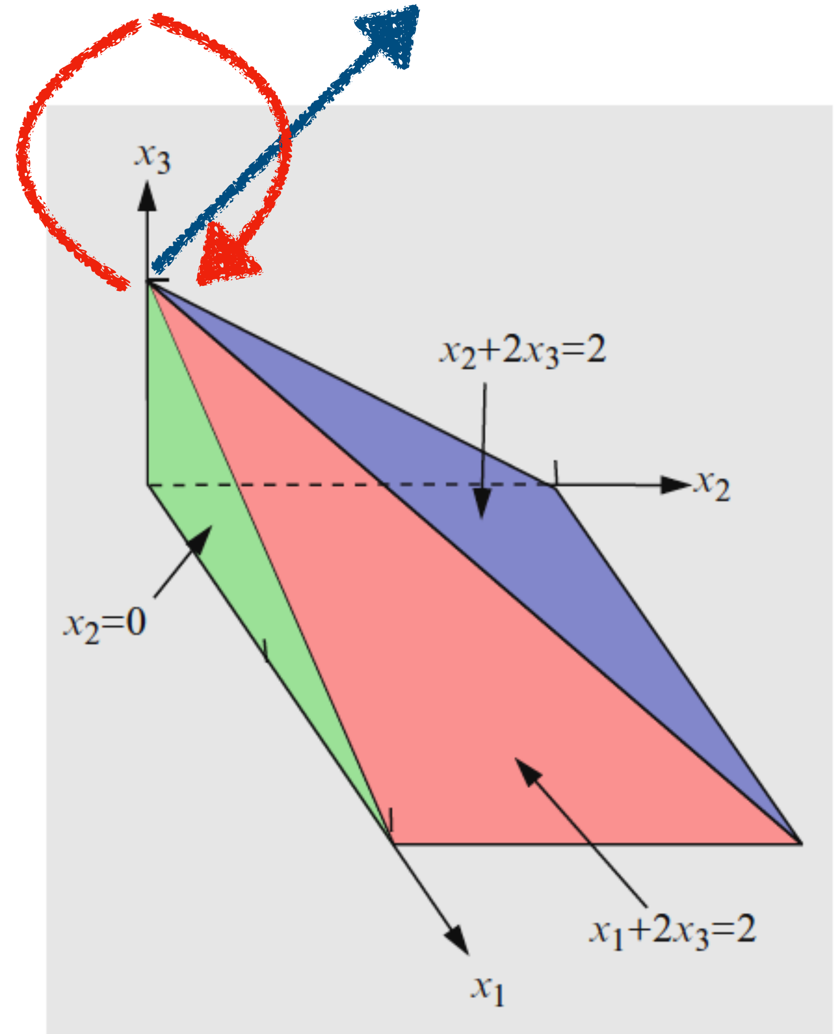

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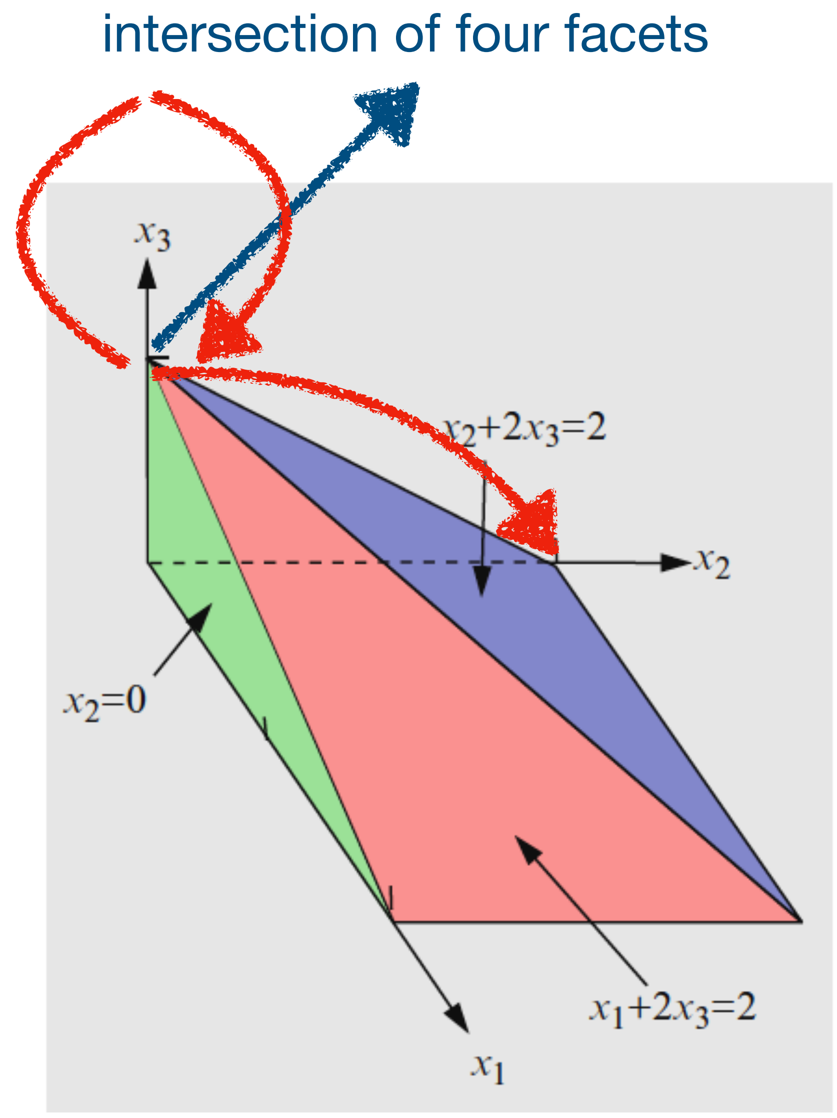

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But it corresponds to four different basic feasible solutions/dictionaries.

Termination

Theorem: If the simplex method does not cycle, it terminates.

Proof: A dictionary is determined by which variables are basic and which are non-basic.

There only $\binom{n+m}{m} = \frac{(n+m)!}{n!m!}$ possibilities.

Simplex Running Time

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More good news: "*Beyond the worst-case analysis*" shows that the algorithm is also efficient in theory.

Even more good news: We have other algorithms that run in worst-case polynomial running time (Ellipsoid Method, Interior Point Methods).

Duality

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Maximisation becomes minimisation.

The two linear programs will have a very important connection.

Duality

The Primal

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1\in}^{n} \alpha_{ij} x_j \leq b_i, \quad i = 1, ..., m$$

$$x_j \geq 0, \quad j = 1, ..., n$$

The Dual

$$\text{minimize} \quad \sum_{i=1}^{m} b_i y_i$$

$$\text{subject to} \quad \sum_{i=1}^{m} \alpha_{ij} y_i \geq c_j, \quad i = 1, ..., n$$

$$y_j \geq 0, \qquad j = 1, ..., m$$

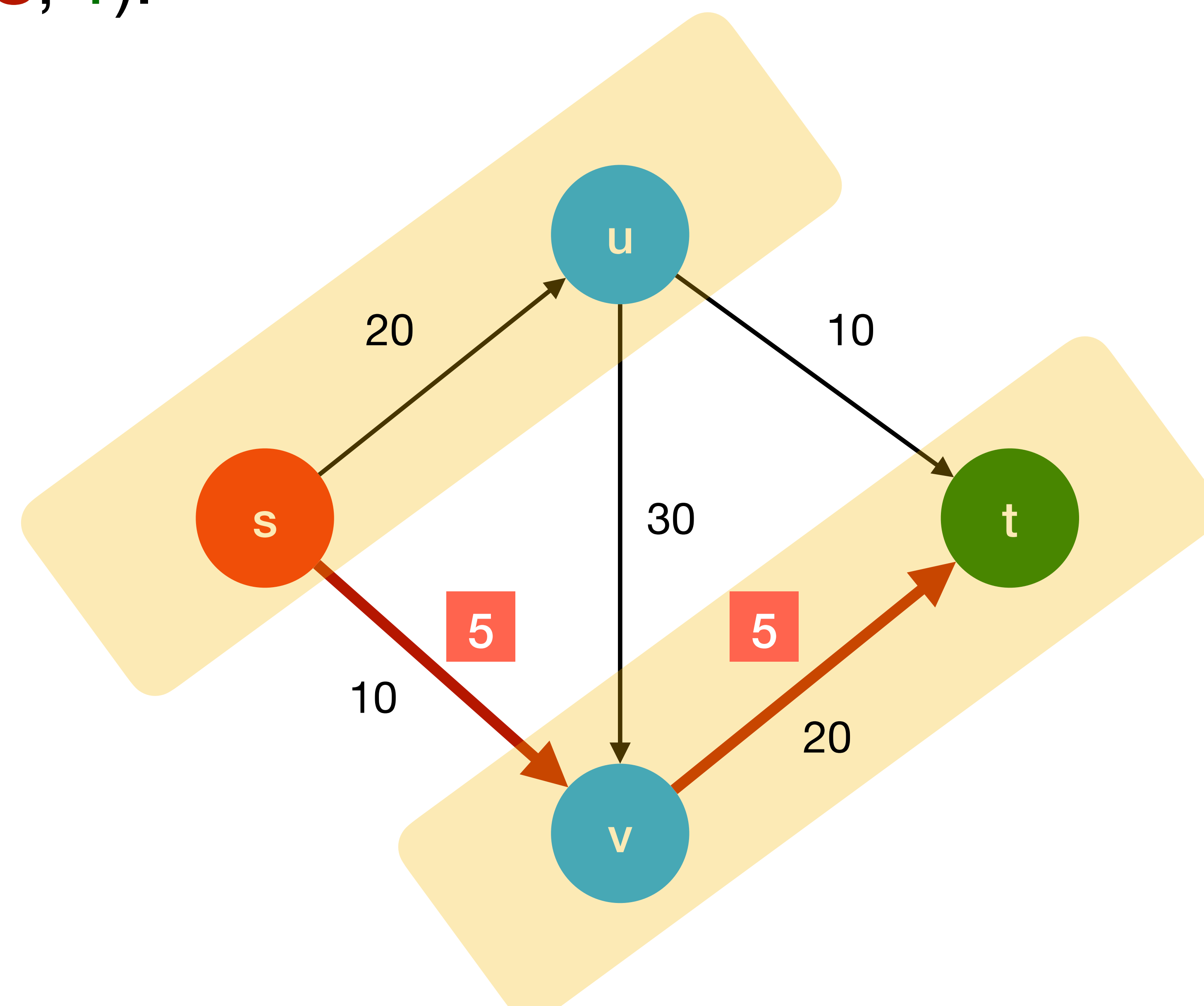
Weak Duality

Let x be any feasible solution to the Primal and let y be any feasible solution to the Dual. Then we have that

$$\text{value}(x) \leq \text{value}(y)$$

Weak Duality

Fact 3: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then v(f) ≤ c(S, T).



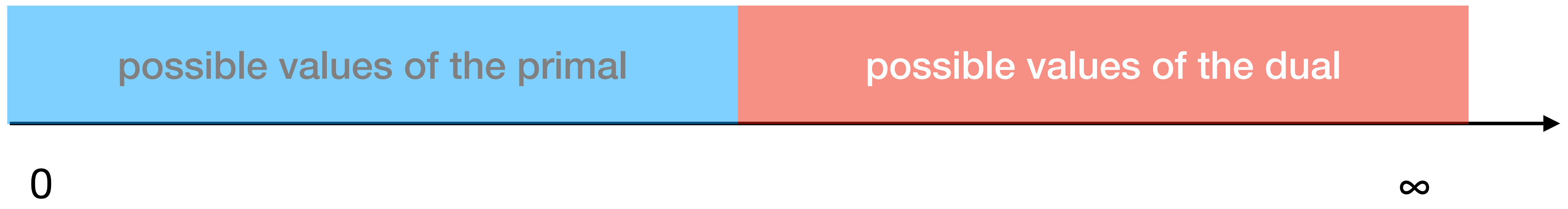
Strong Duality

Let x be any feasible solution to the Primal and let y be any feasible solution to the Dual. If

$$\text{value}(x) = \text{value}(y)$$

then x and y are both optimal solutions.

Pictorially



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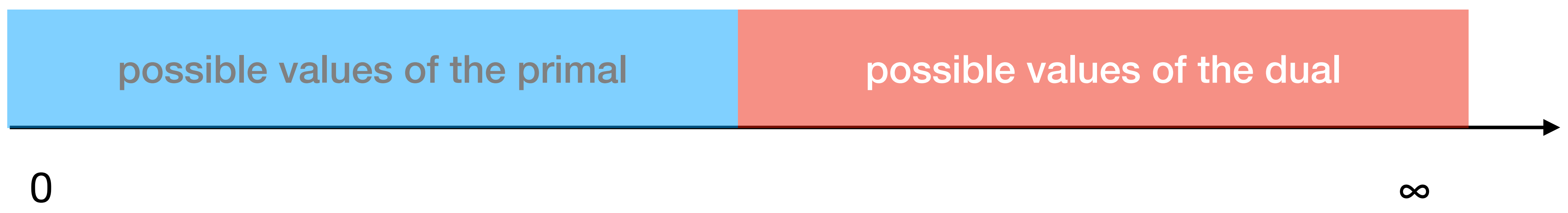

How can we prove that a solution x to the primal is maximum?

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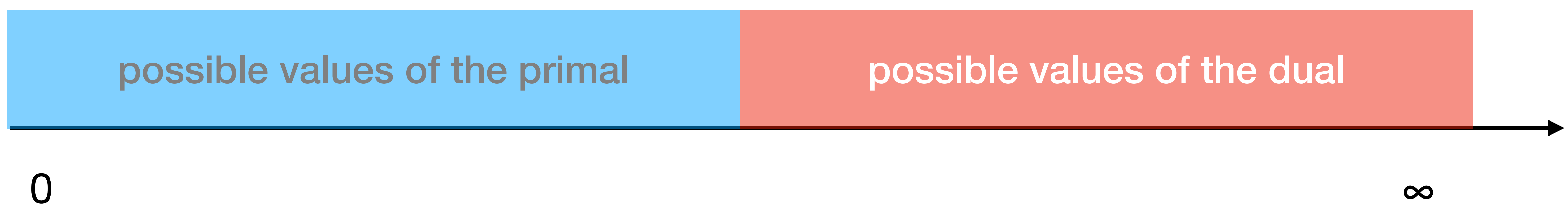

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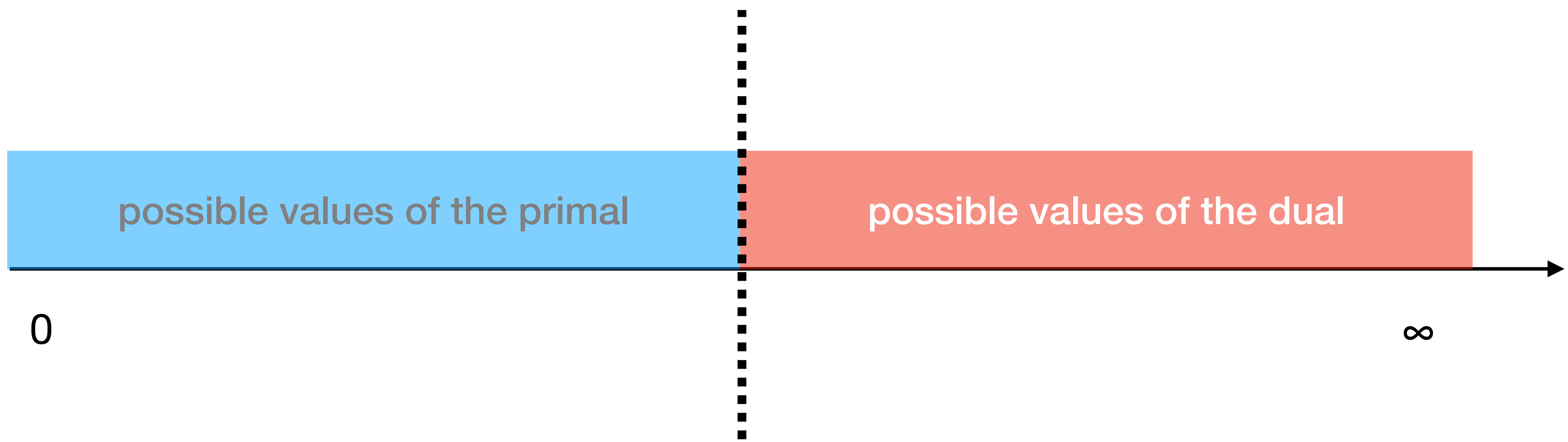

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The Max-Flow Min-Cut Theorem

Theorem: In every flow network, the value of the maximum flow is *equal* to the capacity of the minimum cut.

This is a consequence of the *strong duality theorem* for linear programs!

As a matter of fact…

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To prove that the solution is optimal, we use duality.

Do you know of any other algorithms for which the same principle applies?

Maximum Flow as an LP

We can write the maximum flow problem as a linear problem.

"Maximise the flow, subject to capacity and flow conservation constraints".

$$
\begin{array}{llll}
\text{maximise} & \displaystyle\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} & & \\
\text{subject to} & f_{uv} \le c_{uv}, & \text{for each} & u, v \in V \\
& \displaystyle\sum_{v\in V} f_{vu} = \sum_{v\in V} f_{uv}, & \text{for each } u \in V - \{s, t\} & \\
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total flow out of s total flow into s

$$\text{maximise} \quad \boxed{\sum_{v\in V} f_{sv}} - \boxed{\sum_{v\in V} f_{vs}}$$

capacity constraint

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total flow into s

capacity constraint

flow conservation

non-negative flow

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If u is in S and v is
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LP-relaxation

An *LP-relaxation* of an Integer Linear Program is a linear program which is identical to the ILP, except all the integrality constraints have been removed ("*relaxed*"), or replaced with non-integral constraints.

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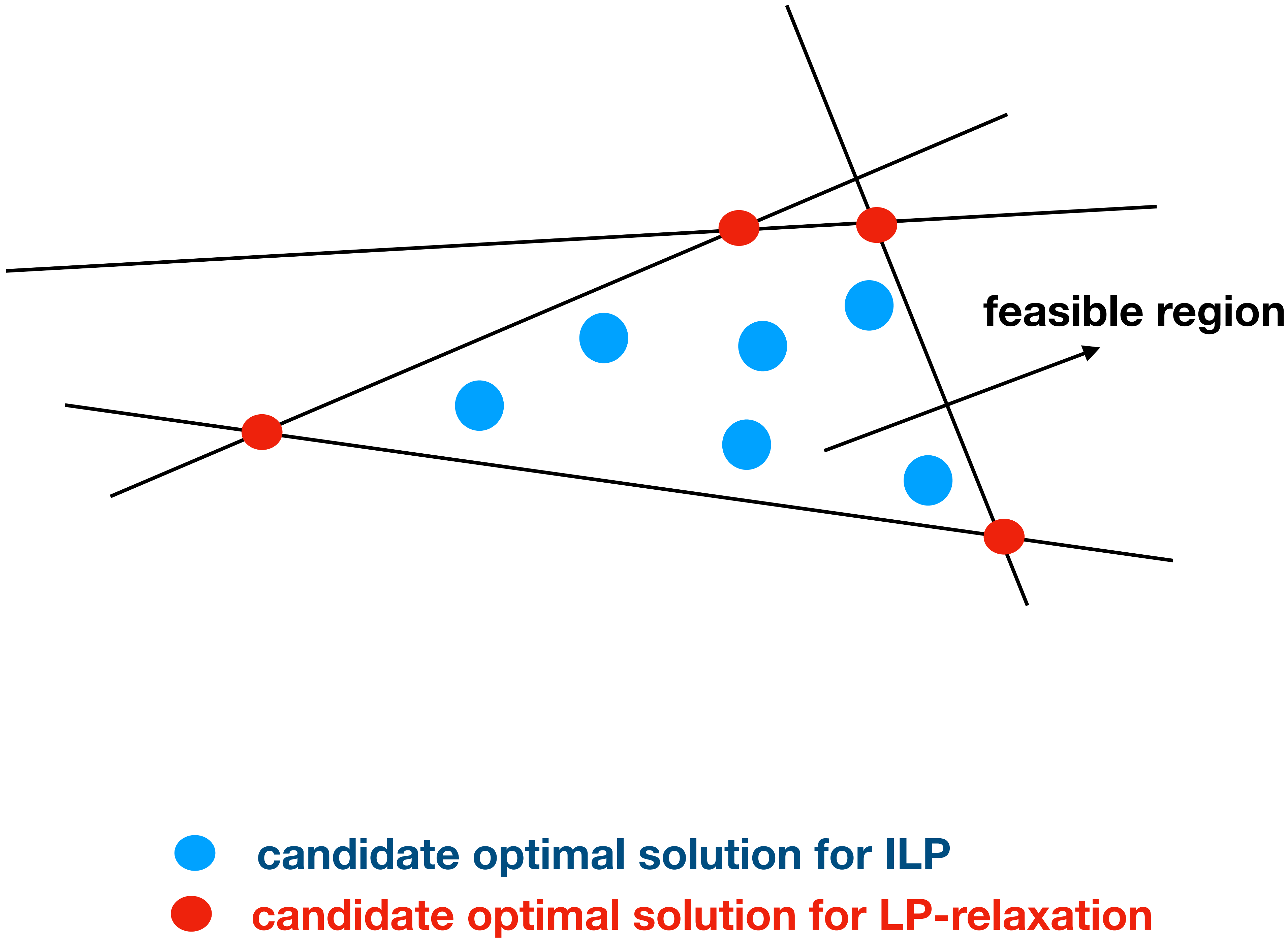
ILP vs LP-relaxation



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The ratio

max_value(LP-relaxation) / max_value(LP)

is called the integrality gap of the LP-formulation.

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In other words, the Min-Cut LP has *an integer optimal solution*.

Back to Maximum Flow

What if we wanted an integer flow instead of any flow?

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Back to Maximum Flow

Does the LP-relaxation of this ILP always have an integer solution?

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For Max-Flow, it finds the maximum fractional flow.

For Min-Cut, it finds the minimum “fractional” cut.

If we solve those LP-relaxations, we will get *integer solutions*.

Totally Unimodular Matrices

Let A be a real *m* x *n* matrix. Suppose that every square submatrix of A has determinant in {*0*, *+1*, *-1*}. Then A is *totally unimodular*.

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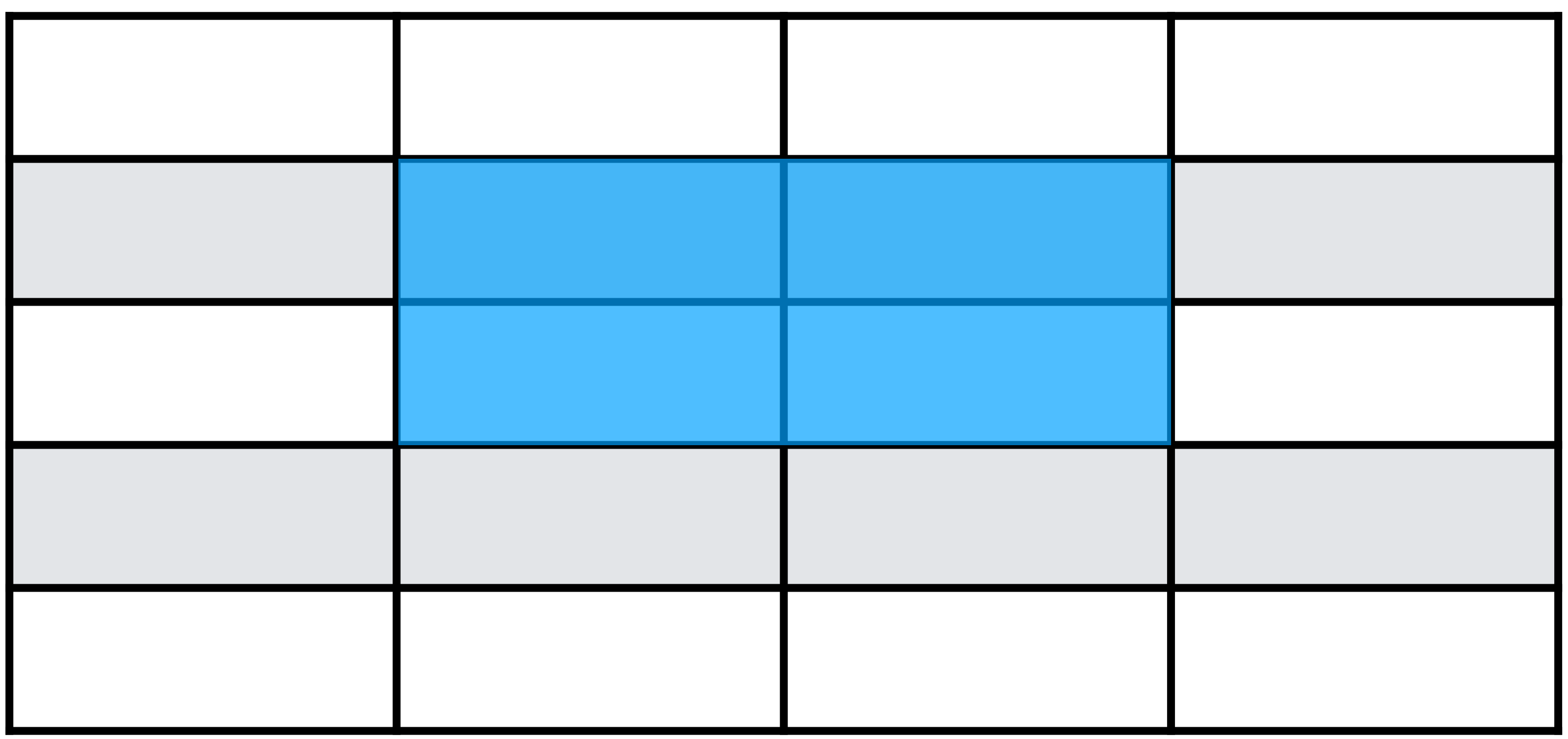

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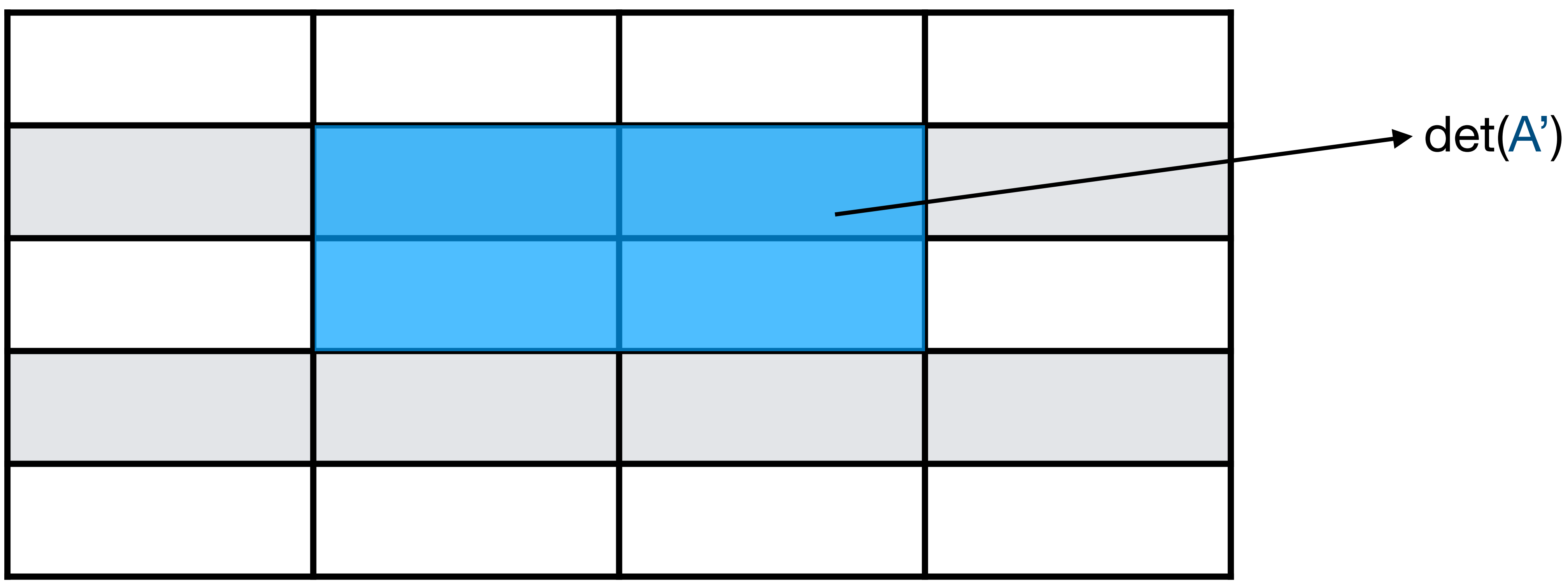

Total Unimodularity

If the constraint matrix A is totally unimodular and b is an *integer vector*, then the LP has an *integer solution*.

The Max-Flow and Min-Cut LP-relaxations admit integer solutions because their constraint matrices are totally unimodular.

$$\begin{aligned} \text{maximise} \quad & c^{\mathrm{T}} x \\ \text{subject to} \quad & Ax \le b, \\ & x \ge 0 \end{aligned}$$

To be more precise

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Lemma: Suppose A is a totally unimodular matrix and b is an integer vector. Then every extreme point of

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Claim: Suppose A is totally unimodular. Then the matrix $A' = (A\ {-A}\ I\ {-I})^T$ is also totally unimodular.

Corollary: Suppose A is a totally unimodular matrix and b is an integer vector. Then every extreme point of

$P = \{x: Ax = b, 0 \leq x \leq c\}$ is integral.

Back to maximum flow

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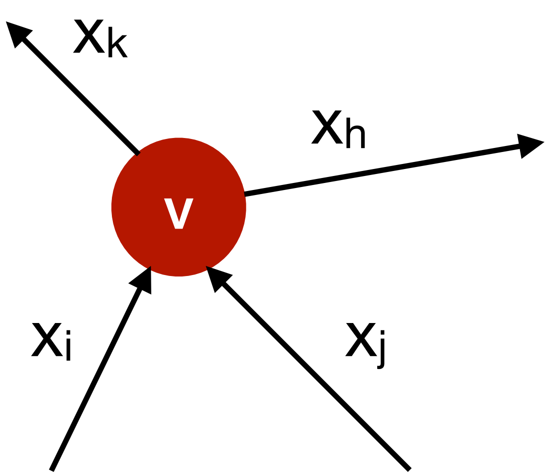

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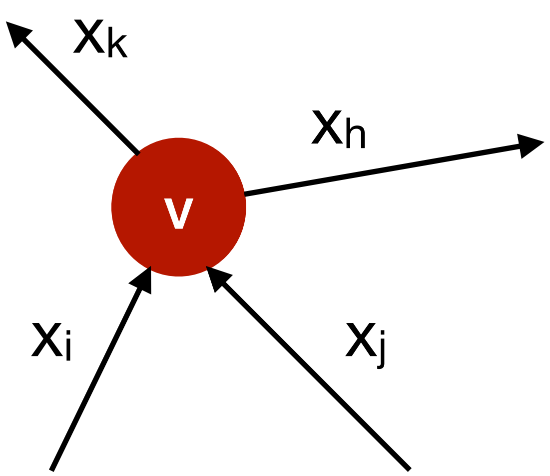


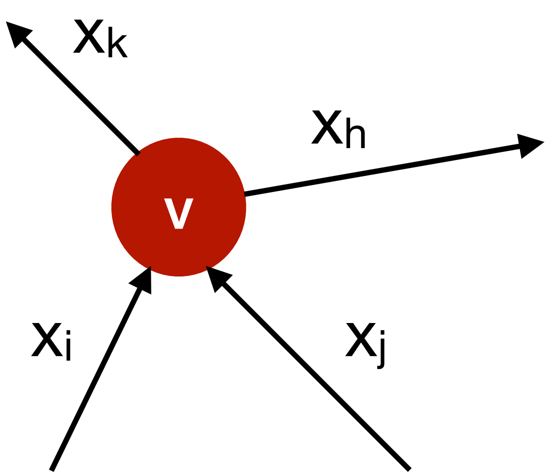
$0 \le x \le c$

$Ax = 0$

$x_k + x_h - x_i - x_j = 0$

Back to maximum flow

$$
\begin{array}{lll}
\text{maximise} & \displaystyle\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs} & \\
\text{subject to} & f_{uv} \le c_{uv}, & \text{for each } u, v \in V \\
& \displaystyle\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}, & \text{for each } u \in V - \{s, t\} \\
& f_{uv} \ge 0, & \text{for each } u, v \in V
\end{array}
$$



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$A_{vi} = A_{vj} = -1$

$A_{vk} = A_{vh} = 1$

Back to maximum flow

- Consider the *incidence matrix* of the flow network (without s and t):
  - $A_{ij} = 1$ if edge j **starts** at node i in G_f.
  - $A_{ij} = -1$ if edge j **ends** at node i in G_f.
  - $A_{ij} = 0$ otherwise.

Nodes/Edges		*j*	*k*
i		1	-1

Back to maximum flow

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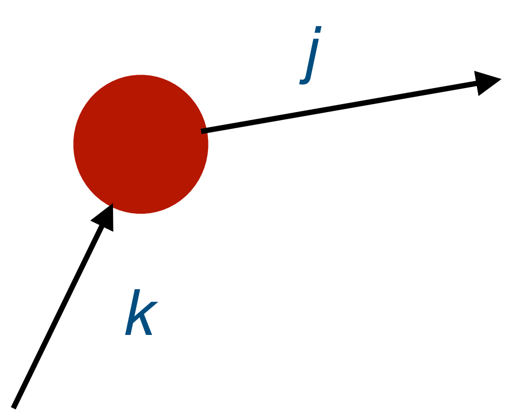

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Back to maximum flow

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This is precisely the matrix A of the max flow LP.

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It suffices to prove that A is totally unimodular, by Corollary.

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Back to maximum flow

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This is precisely the matrix A of the max flow LP.

It suffices to prove that A is totally unimodular, by Corollary.

Lemma: The incidence matrix of any directed graph is totally unimodular.

Nodes/Edges		*j*	*k*
i		1	-1