Algorithms and Data Structures

Asymptotic Notation and Divide and Conquer Fundamentals

Example: Running Time of InsertionSort

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INSERTION_SORT (A)

FOR j ← 2 TO length[A]                          n times
    DO  key ← A[j]                               n-1 times
        {Put A[j] into the sorted sequence A[1 . . j − 1]}
        i ← j − 1                                n-1 times
        WHILE i > 0 and A[i] > key               Σ_{j=2}^{n} t_j times
            DO A[i +1] ← A[i]                    Σ_{j=2}^{n} (t_j − 1) times
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for loops, the tests are executed one more time than the loop body

$$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

Best case? **Sorted array,** $t_j = 1$ Bounded by some cn for some constant c

Worst case? **Reverse sorted array,** $t_j = j$ Bounded by some cn^2 for some constant c

Asymptotic Notation

- When n becomes large, it makes less of a difference if an algorithm takes $2n$ or $3n$ steps to finish.
- In particular, $3 \lg n$ steps are fewer than $2n$ steps.
- We would like to avoid having to calculate the precise constants.
- We use asymptotic notation.

Asymptotic Notation

O-notation. $O(g(n)) = f(n)$: there exist positive constants c and n_0 such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

Ω-notation. $\Omega(g(n)) = f(n)$: there exist positive constants c and n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$.

Θ-notation. $\Theta(g(n)) = f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.

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"The rate of growth of $f(n)$ is at most that of $g(n)$."

Asymptotic Notation

O-notation. $O(g(n)) = f(n)$: there exist positive constants c and n_0 such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

For sufficiently large inputs, there is a constant such that $c \cdot g(n)$ is not smaller than $f(n)$.

For example, for sufficiently large inputs, $2n$ is larger than $3 \lg n$. Therefore, $3 \lg n = O(n)$.

Use: If we can upper bound the running time of an algorithm by $c \cdot g(n)$, where c is some constant and $g(\,\cdot\,)$ is a function of the input, then we can say that the running time is $O(g(n))$.

Asymptotic Notation

Ω-notation. $\Omega(g(n)) = f(n)$: there exist positive constants c and n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$.

"The rate of growth of $f(n)$ is at least that of $g(n)$."

Θ-notation. $\Theta(g(n)) = f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$. "The rate of growth of $f(n)$ is at most that of $g(n)$."

"The rate of growth of $f(n)$ is the same as that of $g(n)$."

Little-O, Little-Omega

o-notation. $o(g(n)) = f(n)$: for any constant c, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < c \cdot g(n)$ for all $n \geq n_0$.

"The rate of growth of $f(n)$ is smaller than that of $g(n)$."

ω-notation. $\omega(g(n)) = f(n)$: for any constant c, there exists a constant $n_0 > 0$ such that $0 \leq c \cdot g(n) < f(n)$ for all $n \geq n_0$.

"The rate of growth of $f(n)$ is larger than that of $g(n)$."

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o-notation. $o(g(n)) = f(n)$: for any constant c, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < c \cdot g(n)$ for all $n \geq n_0$.

Equivalent (but less formal) definition: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

As n approaches infinity, $f(n)$ becomes insignificant compared to $g(n)$.

Example: $2n = o(n^2)$.

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Equivalent (but less formal) definition: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

As n approaches infinity, $g(n)$ becomes insignificant compared to $f(n)$.

Example: $4n^2 = \omega(n)$.

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$5n^3 + 100 = O(n^3)$

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$\lg(n^4) = 4 \lg n = O(\lg n)$

$(4n)^3 = 64n^3 = \Theta(n^3)$

In class quiz

Running time hierarchy

$O(\log n)$	$O(n)$	$O(n \log n)$	$O(n^2)$	$O(n^\alpha)$	$O(c^n)$
logarithmic	linear		quadratic	polynomial	exponential
The algorithm does not even read the whole input.	The algorithm accesses the input only a constant number of times.	The algorithm splits the inputs into two pieces of similar size, solves each part and merges the solutions.	The algorithm considers pairs of elements.	The algorithm performs many nested loops.	The algorithm considers many subsets of the input elements.

constant	$O(1)$	superlinear	$\omega(n)$
superconstant	$\omega(1)$	superpolynomial	$\omega(n^\alpha)$
sublinear	$o(n)$	subexponential	$o(c^n)$

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        i ← j − 1                               n-1 times
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              DO A[i +1] ← A[i]                 Σ_{j=2}^{n} (t_j − 1)  times
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for loops, the tests are executed one more time than the loop body

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A bit more formally:

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(lines 6 and 7: $\sum_{j=2}^{n}(t_j - 1)$ **times**)

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$$T(n) \leq C \cdot n + C' \cdot \frac{n(n+1)}{2} = O(n^2)$$

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- We proved that on any possible input, **InsertionSort** takes time $O(n^2)$.
- This is an **upper bound**, because the running time cannot be more than this (asymptotically).
- Sometimes we can be happy and stop there.
- But what if our analysis was very "loose"?
  - We bounded $t_j \leq j$. Is this possible for this to happen or are we being too "generous"?

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When $g_1(n) = g_2(n)$, we say that our running time analysis is *tight*, and we have fully understood the (asymptotic, worst-case) running time of the algorithm.

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To show the lower bound, we construct explicitly a reverse sorted array (choosing numbers) and explain how the algorithm will make j comparisons in each step j.

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Try it at home!

Upper and Lower (Worst-Case) Bounds

Upper Bound $O(g_1(n))$: On *any possible input* to the problem, our algorithm will take time (at most) $O(g_1(n))$.

Lower Bound $\Omega(g_2(n))$: There *exists at least one input* to the problem, on which our algorithm will take time (at least) $\Omega(g_2(n))$.

When $g_1(n) = g_2(n)$, we say that our running time analysis is *tight*, and we have fully understood the (asymptotic, worst-case) running time of the algorithm.

Introduction to Divide and Conquer

Merging two sorted arrays

Given two sorted arrays **A**[*1,…,n*] and **B**[*1,…,m*], produce a sorted array **C**[*1, …, n+m*] containing all the elements of **A** and **B**.

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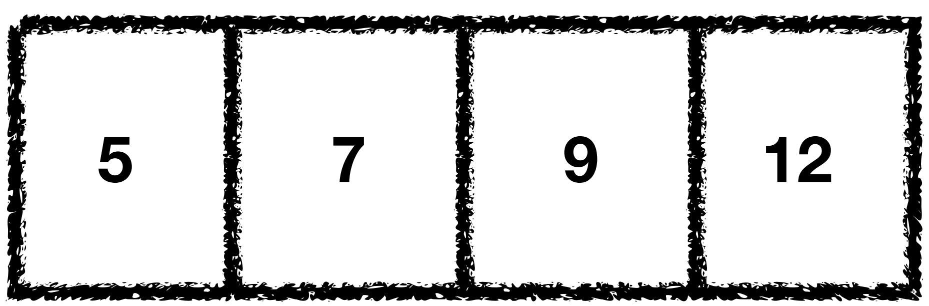

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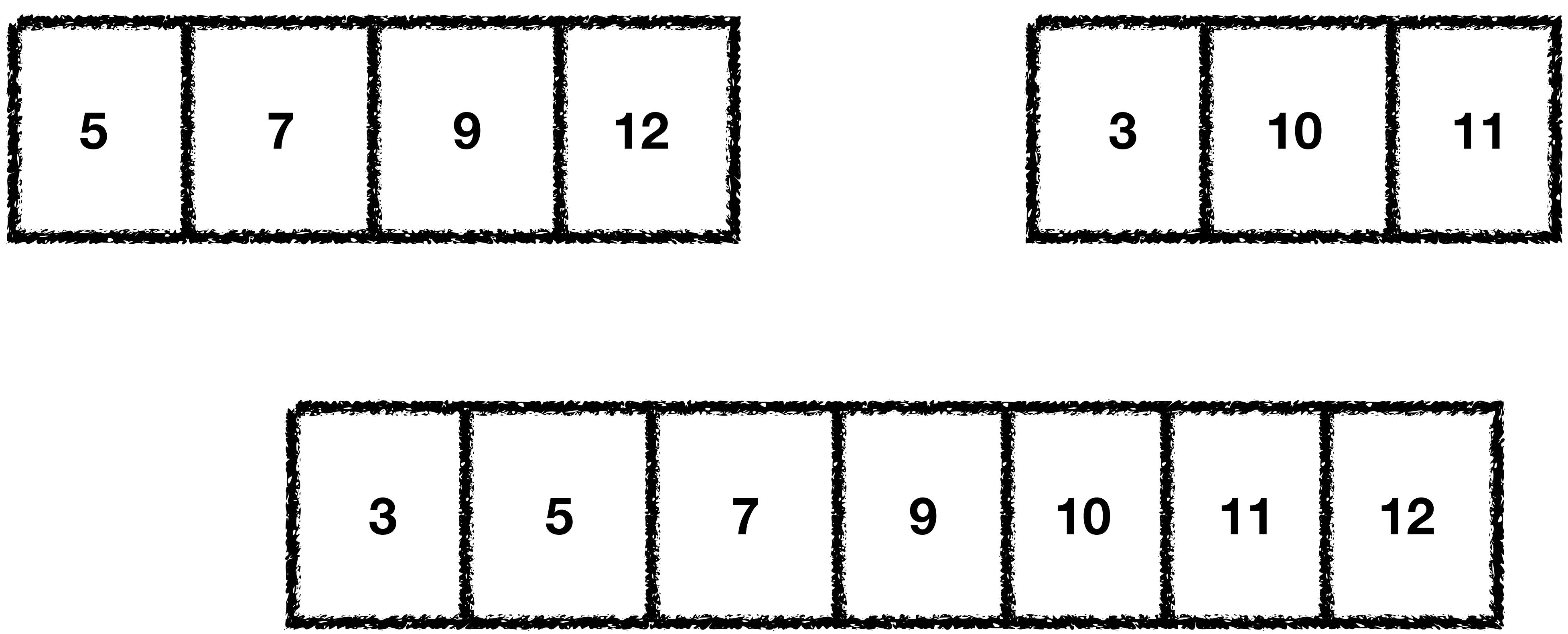

How would you do this?

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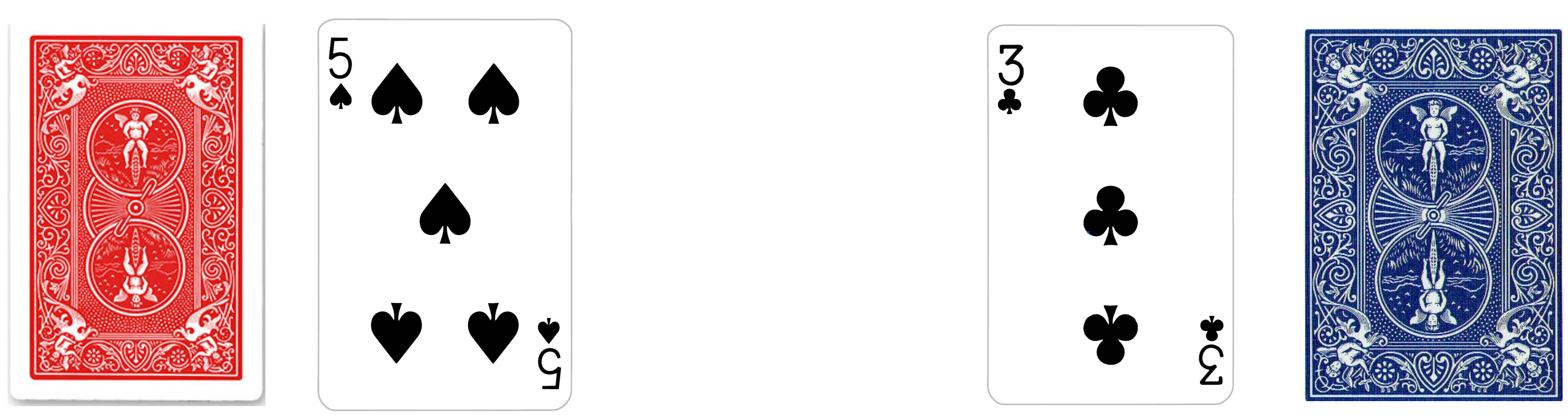

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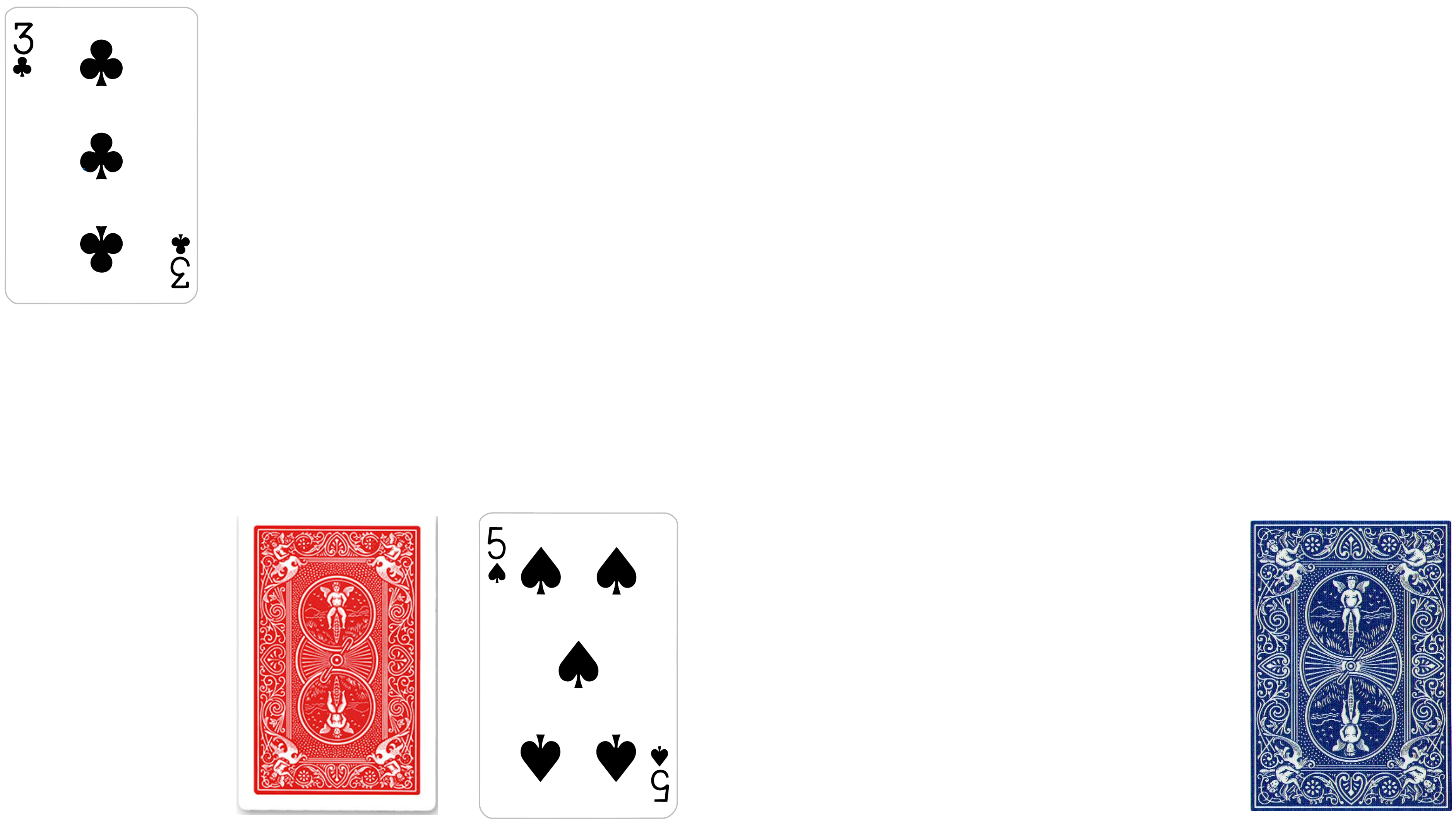

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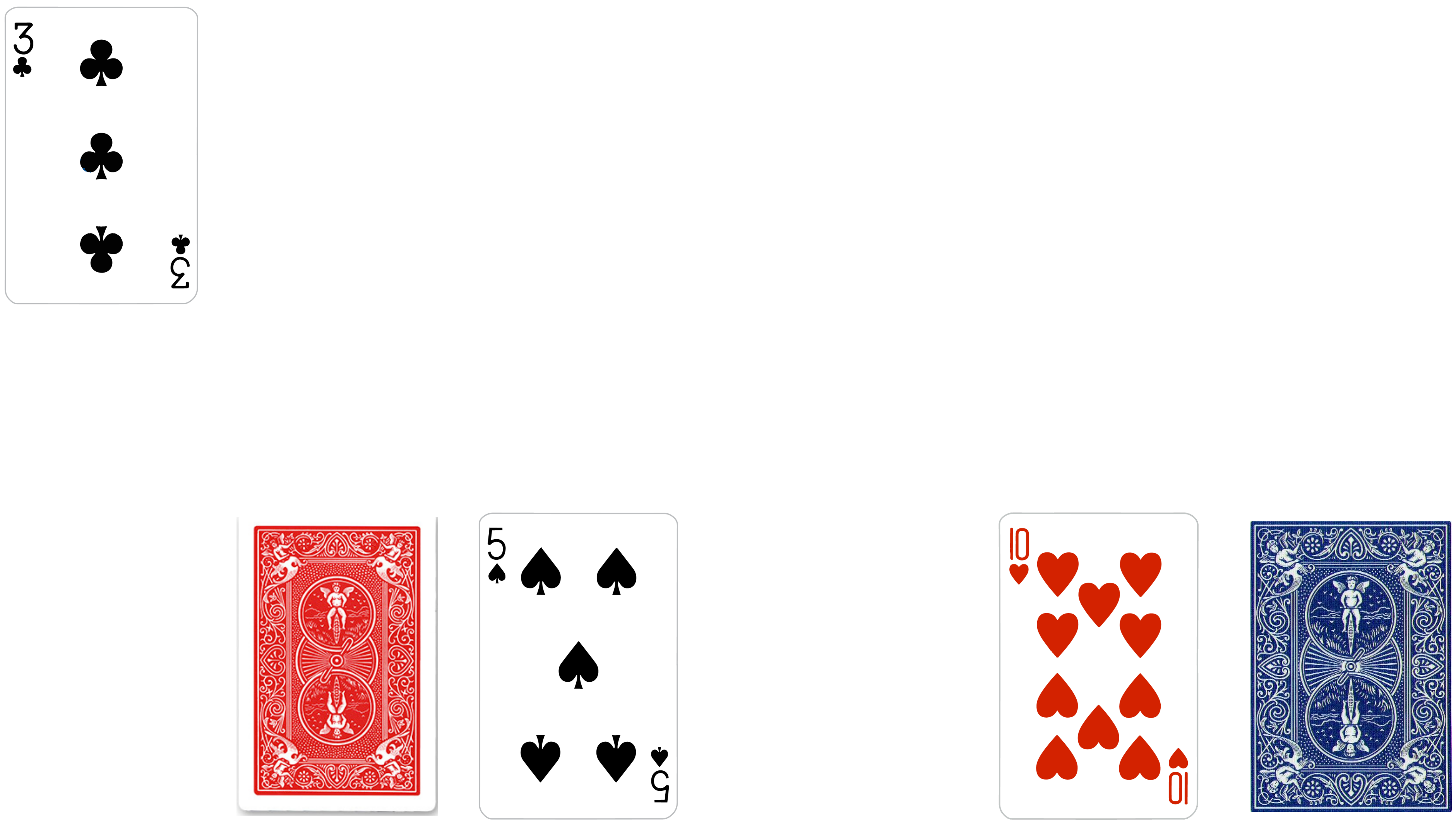

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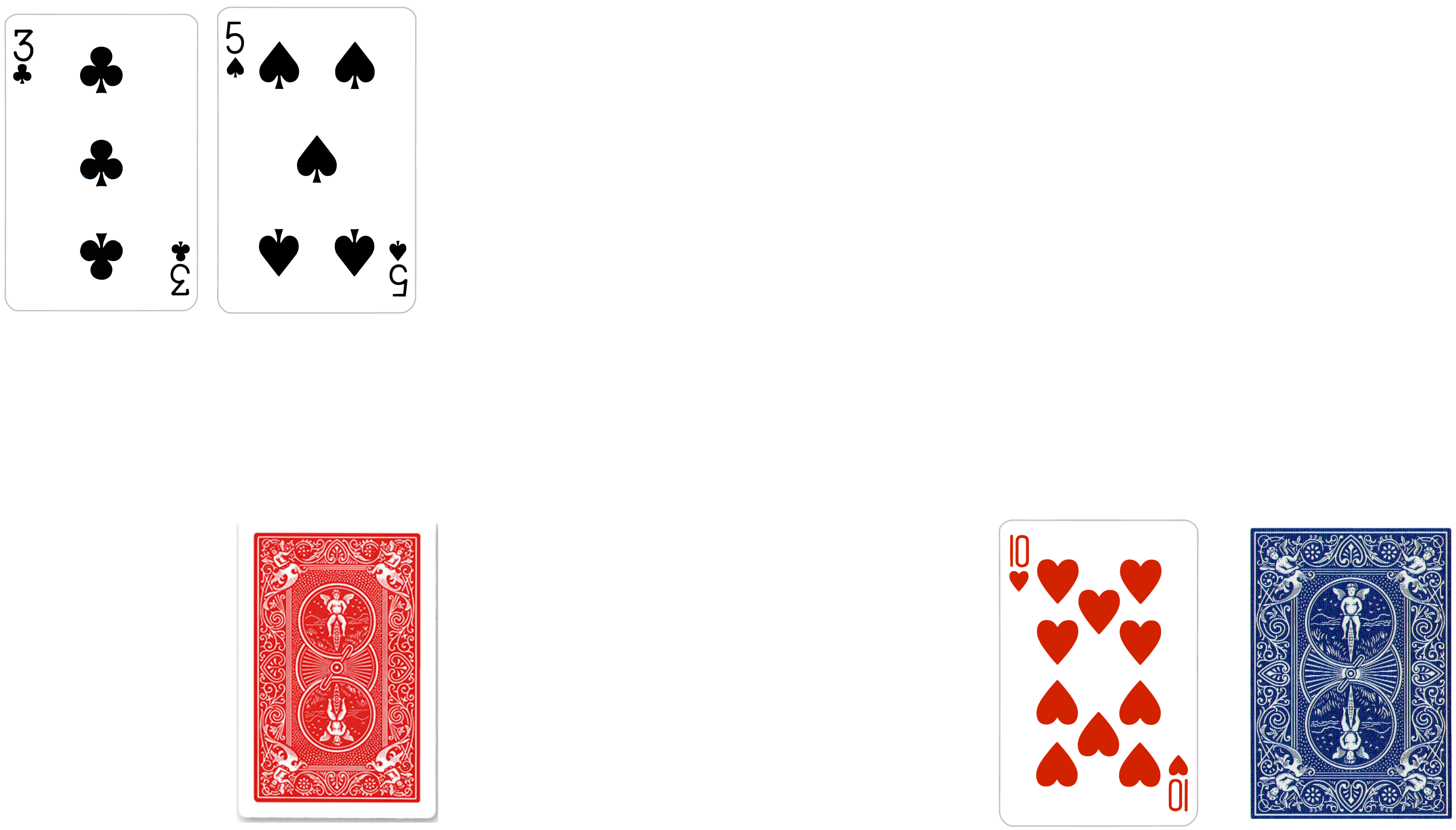

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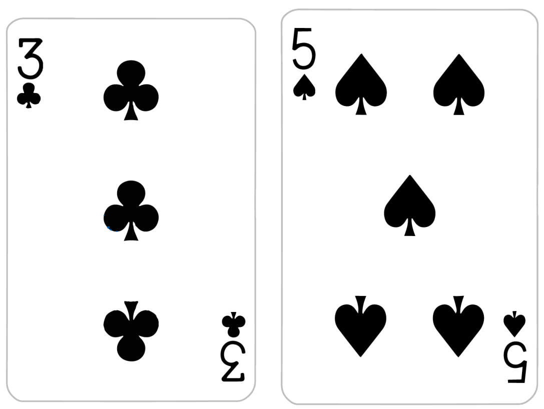

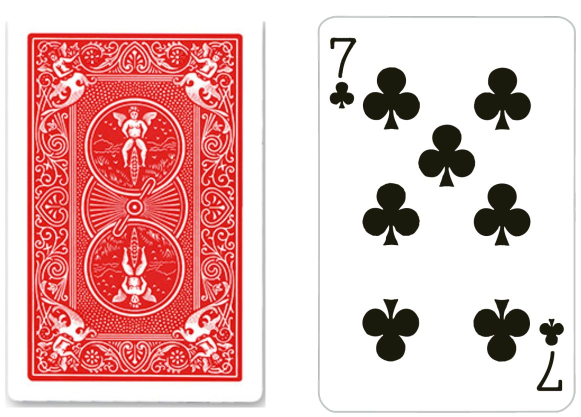

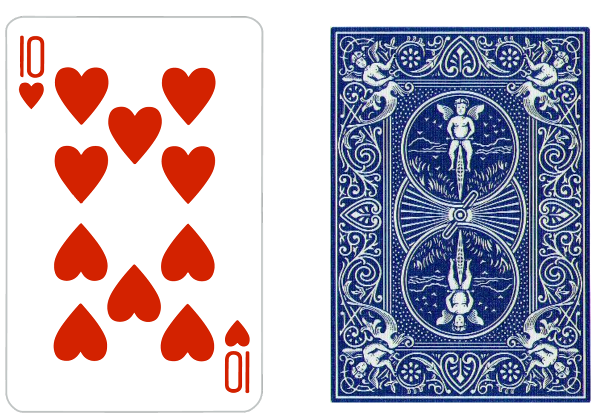

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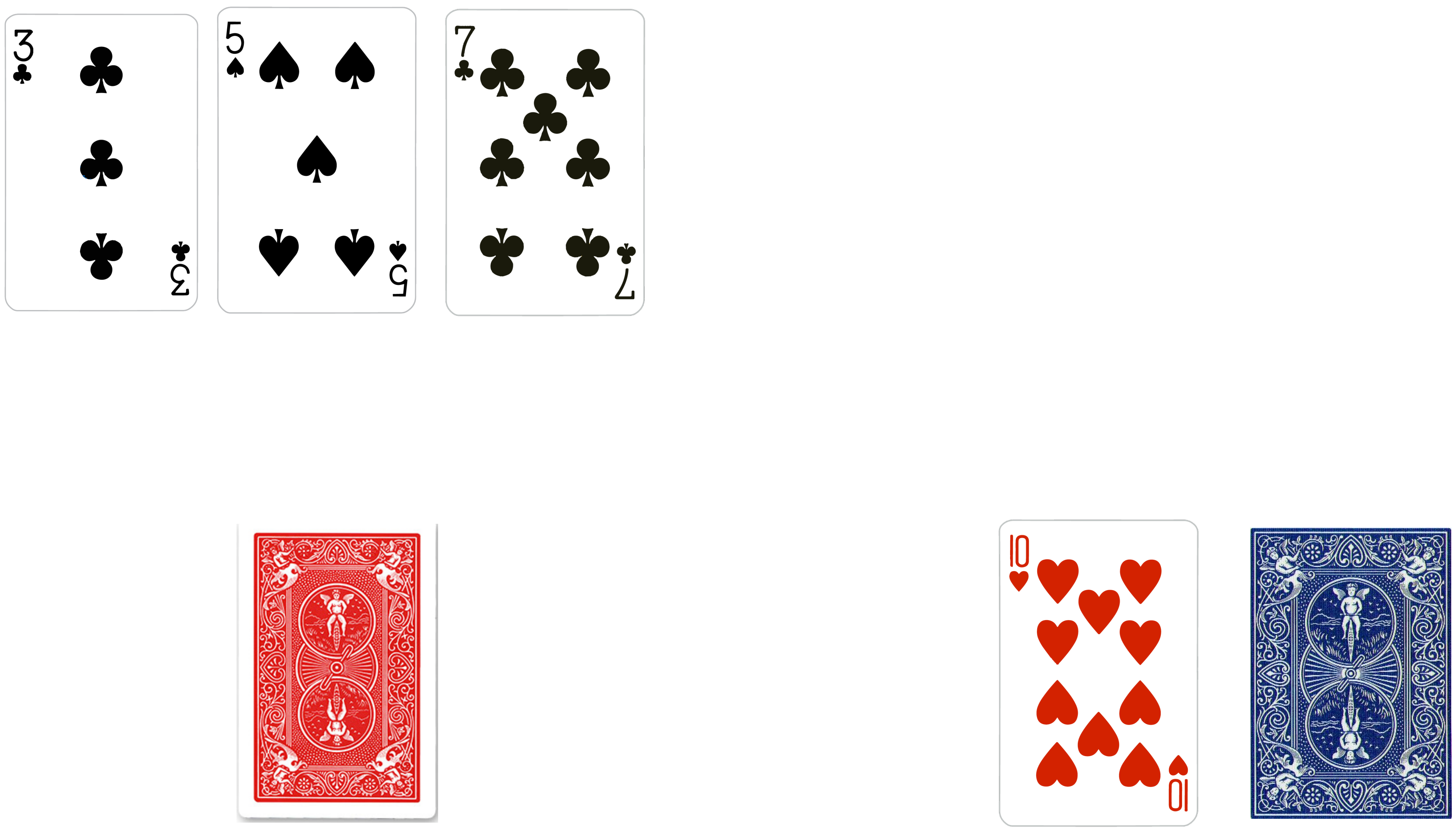

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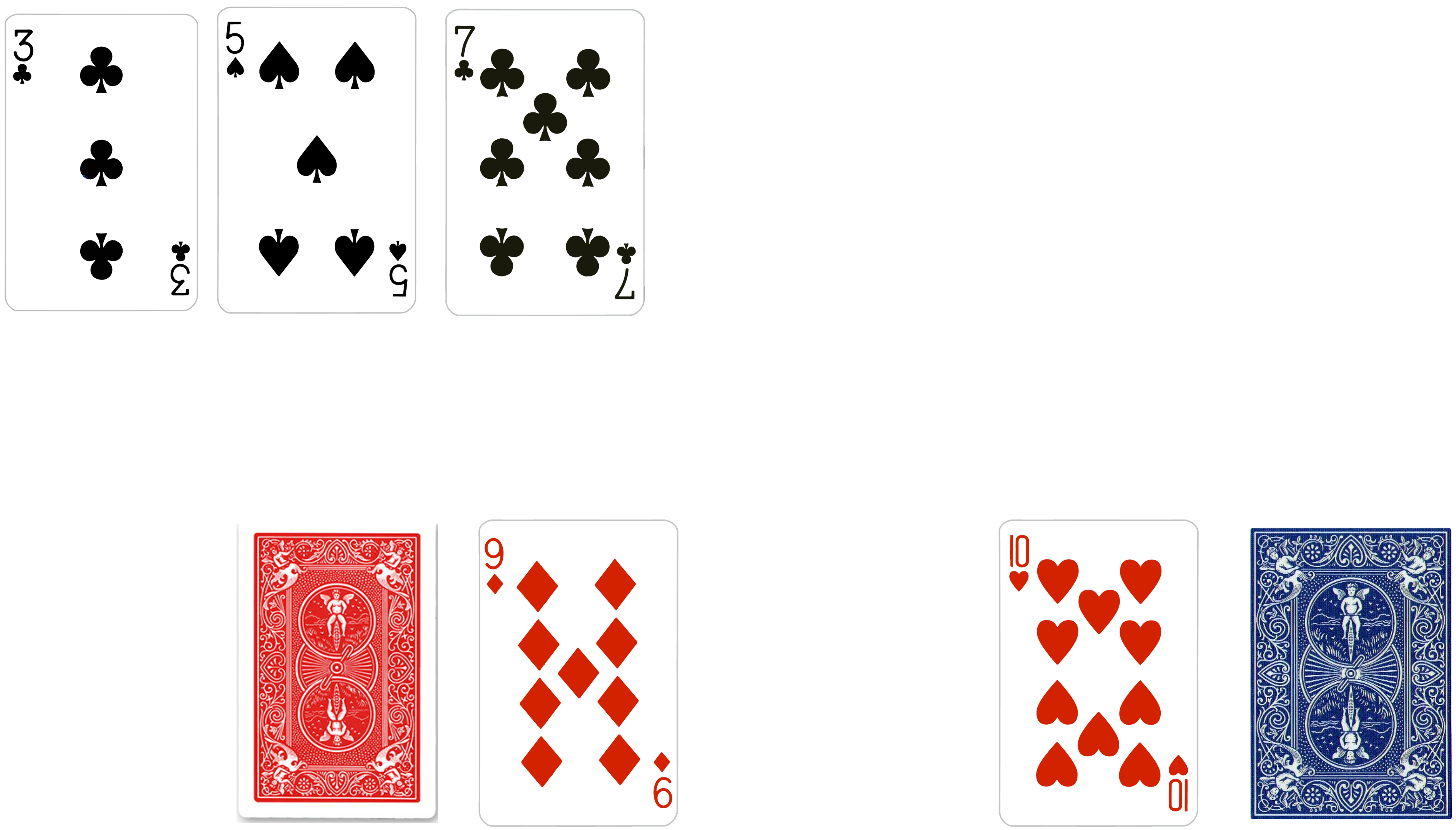

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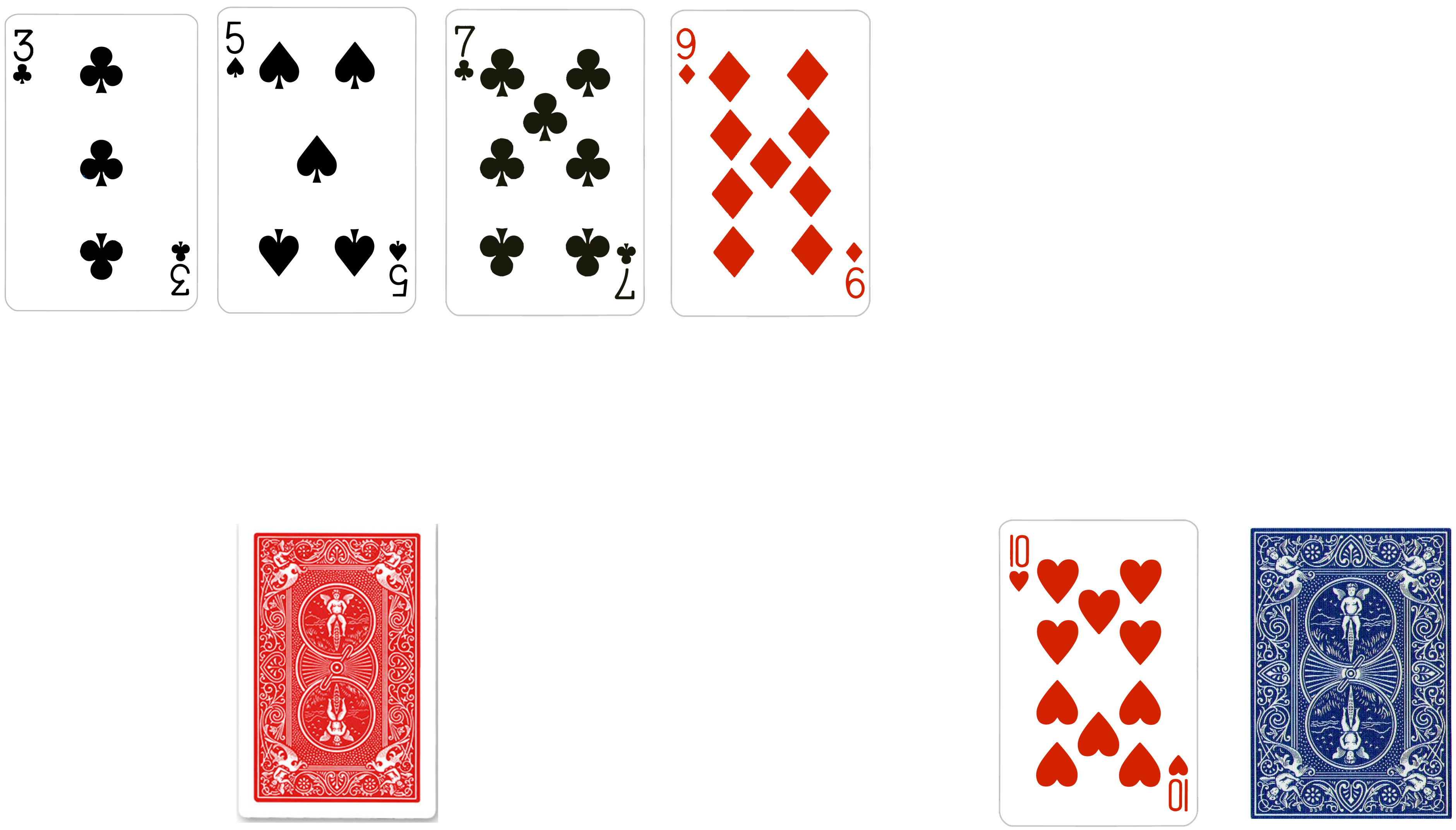

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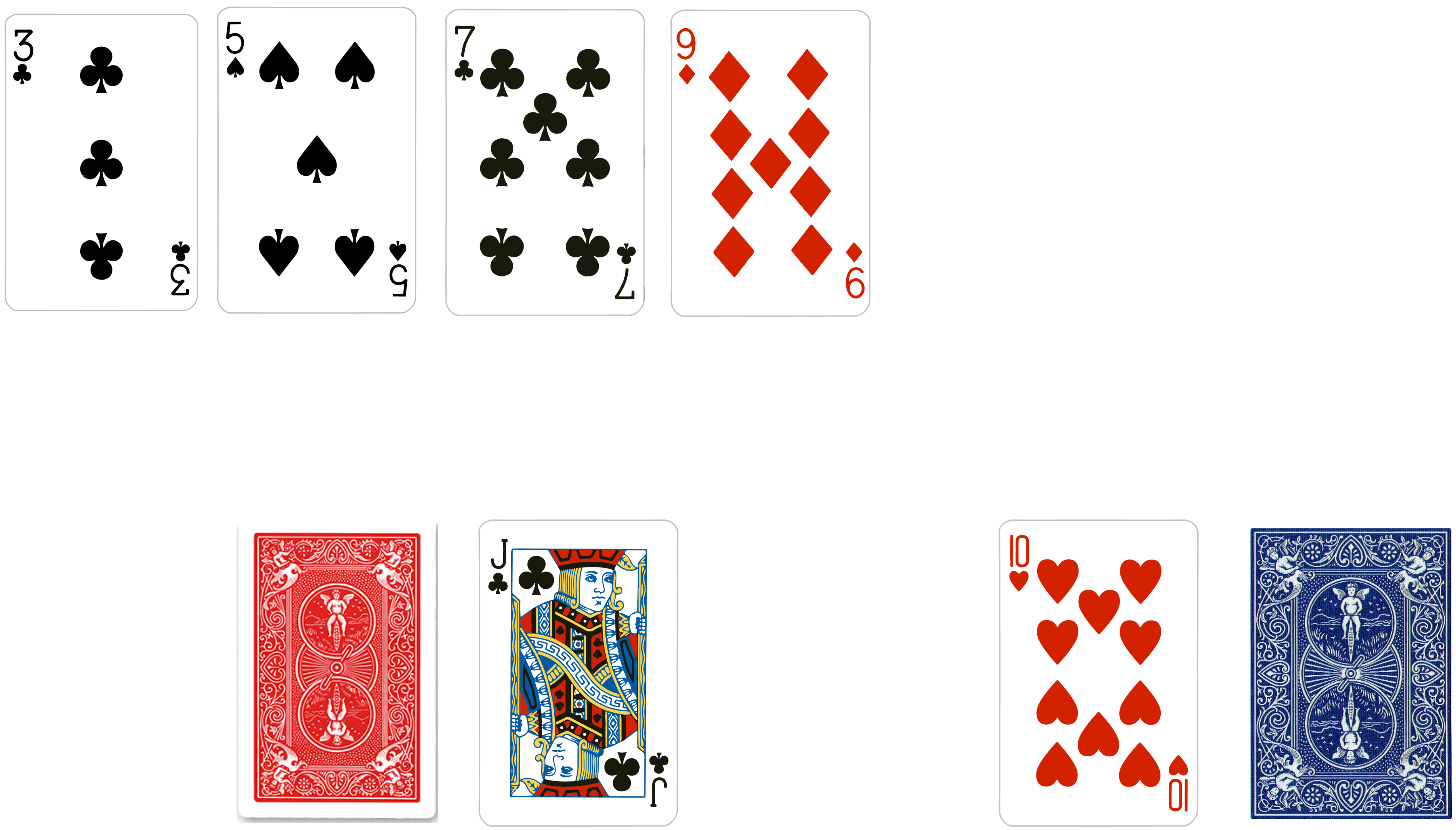

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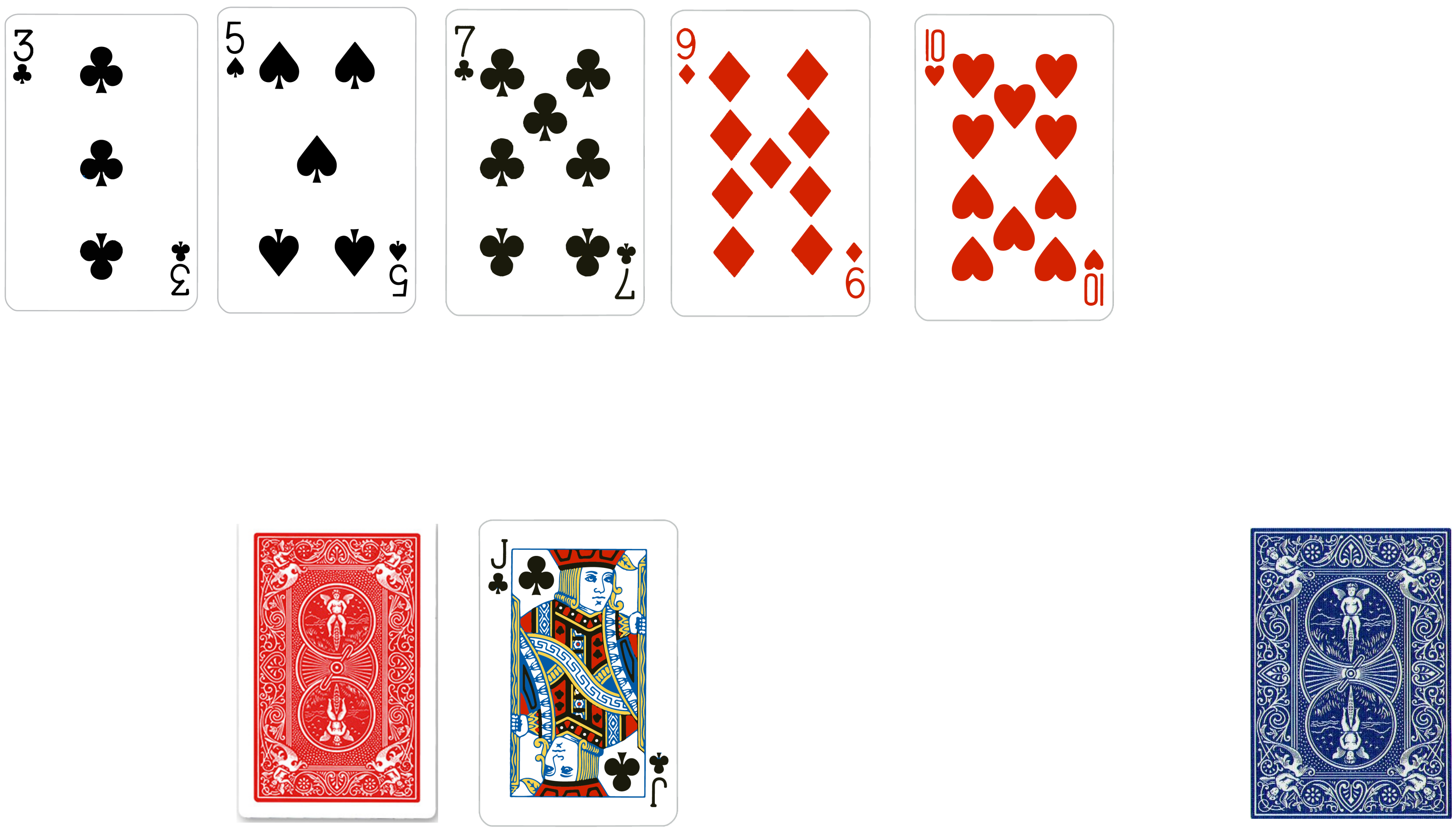

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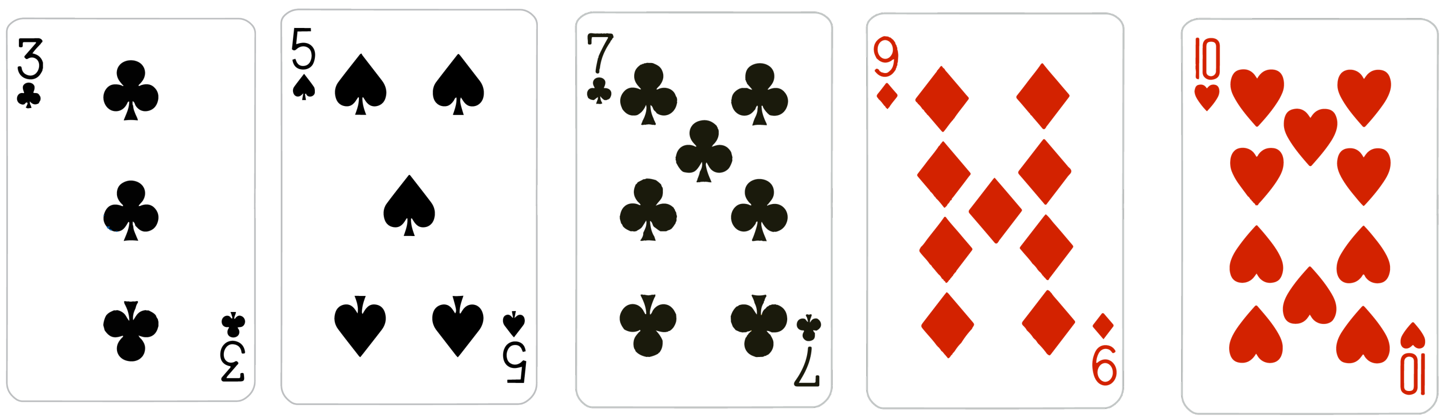

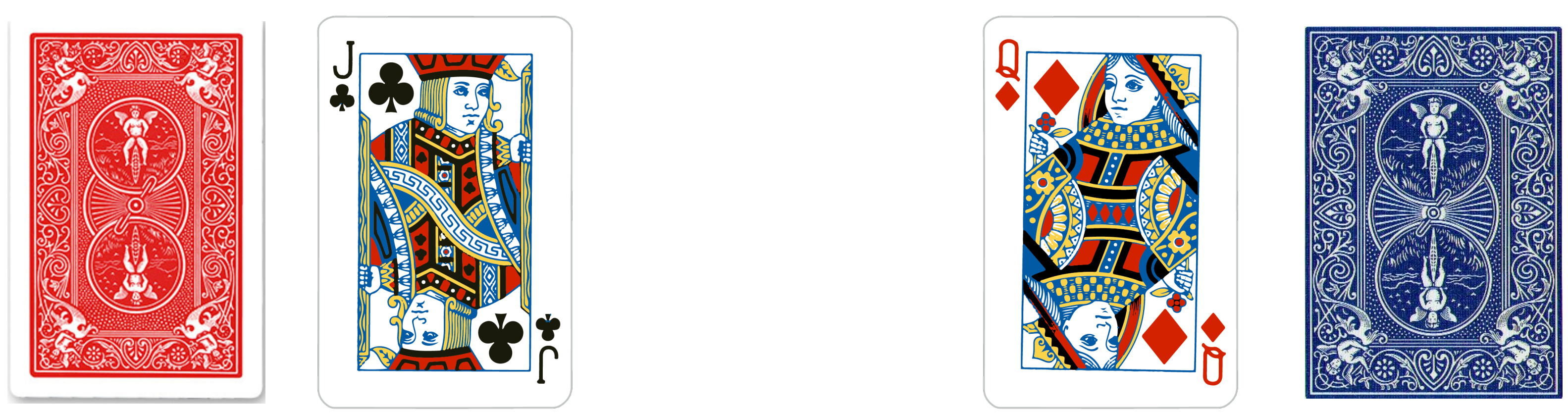

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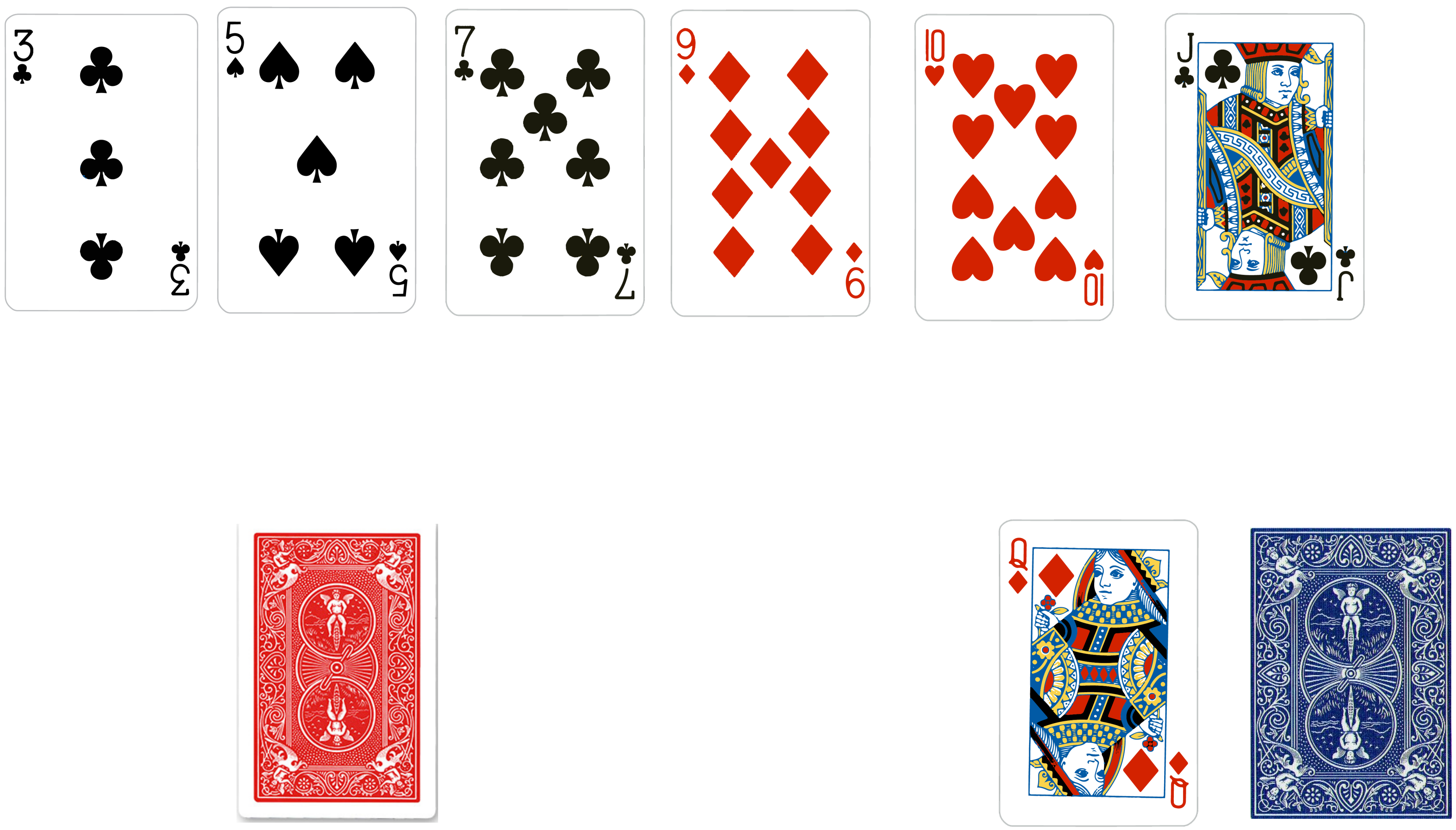

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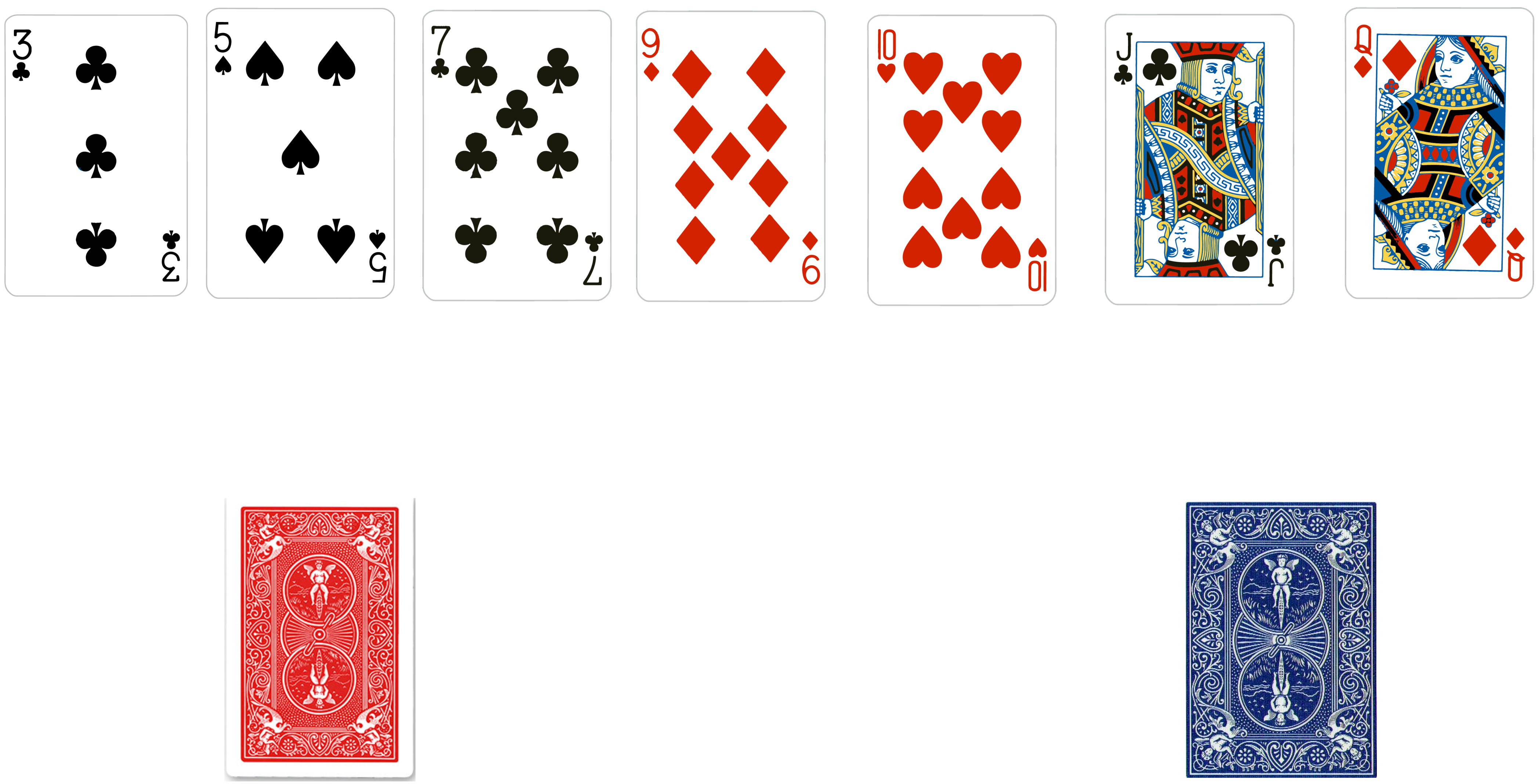

Procedure Merge

Procedure **Merge**(**A, B**)

/* Recall that |**A**| = n and |**B**| = m ***/**

Initialise array **C** of size n+m

i=1, j=1

For k=1, … , m+n-1

    If **A**[i] ≤ **B**[j]
        C[k] = **A**[i]
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i=1, *j*=1

For *k*=1, … , m+n-1

If **A**[*i*] ≤ **B**[*j*]
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O(m+n)

The Mergesort algorithm

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- Merge the sorted subarrays **A**[*1,…,n/2*] and **A**[*n/2+1, …, n*] using the **Merge** procedure.

Mergesort pseudocode

Algorithm **Mergesort**(**A**[$i,\ldots,j$])

If $i=j$, return i

$q=(i+j)/2$

$\mathbf{A}_{left}$=**Mergesort**(**A**[$i,\ldots,q$])

$\mathbf{A}_{right}$=**Mergesort**(**A**[$q+1,\ldots,n$])

return **Merge**($\mathbf{A}_{left}$, $\mathbf{A}_{right}$)

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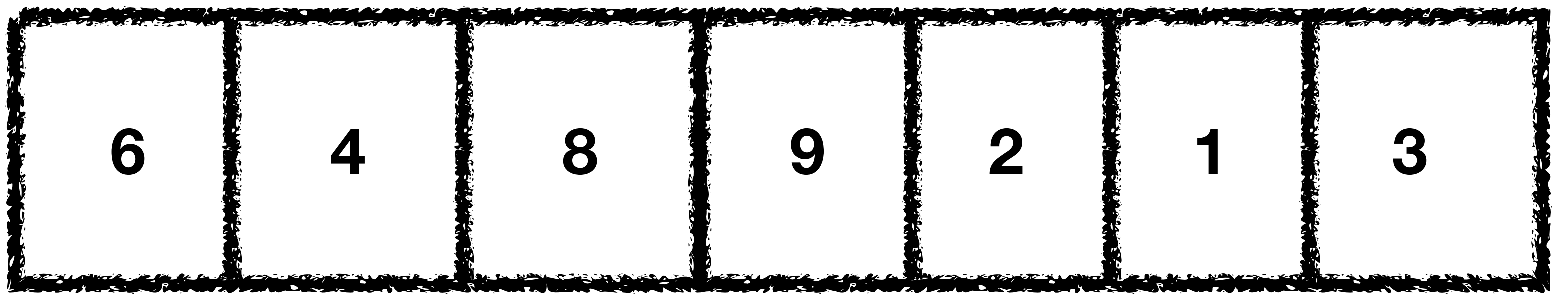
Initial call: **Mergesort**(**A**[$i,...,n$])

Mergesort example

6	4	8	9	2	1	3

Mergesort example

divide



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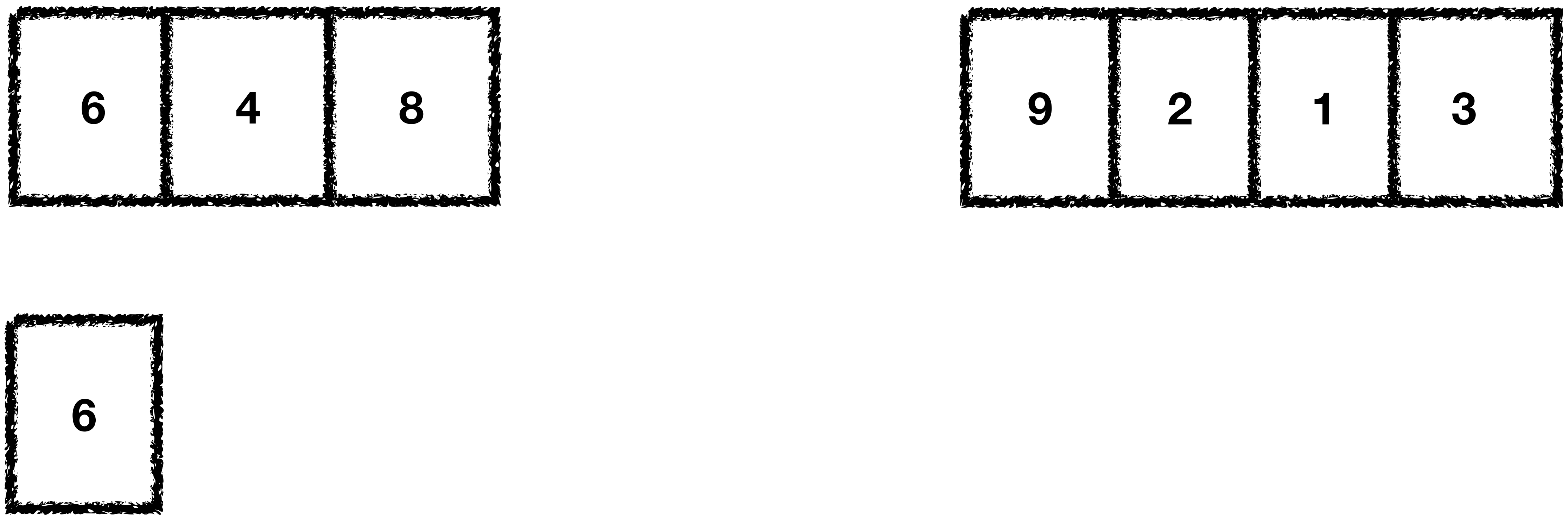

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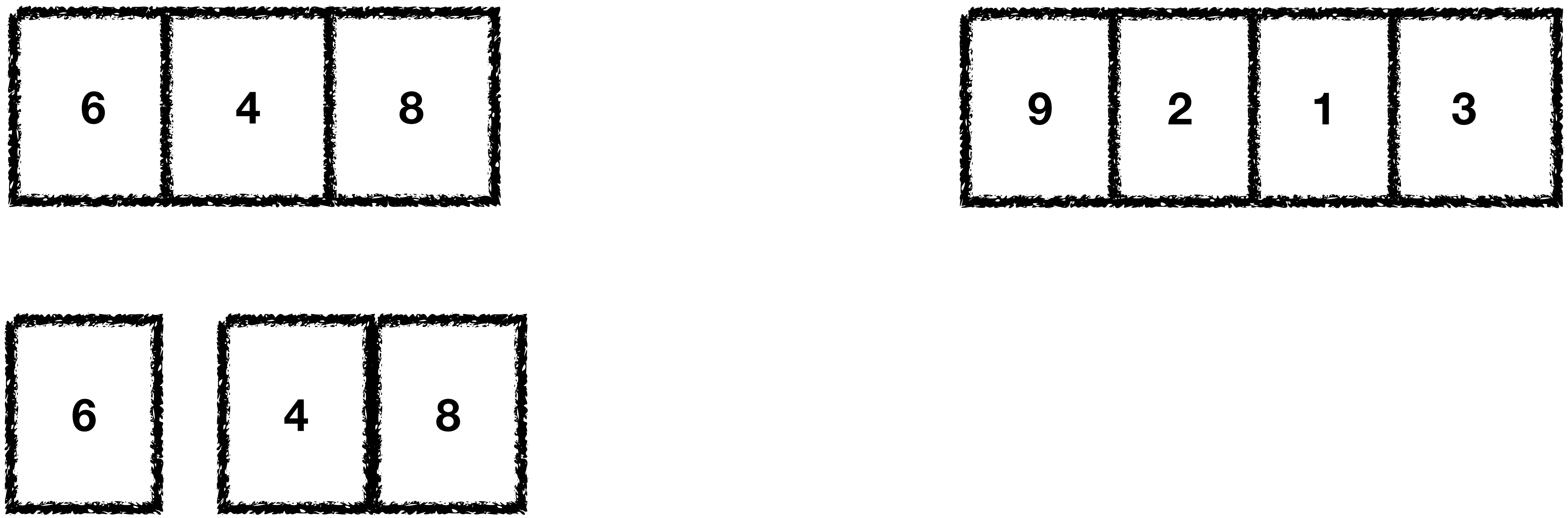

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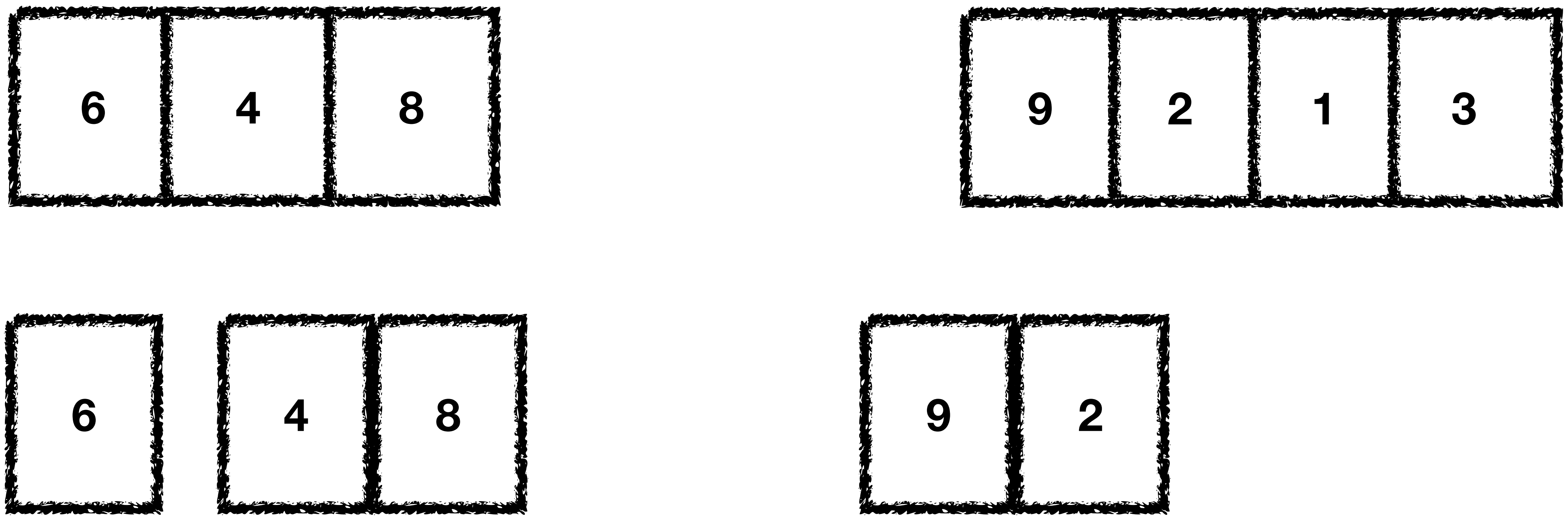

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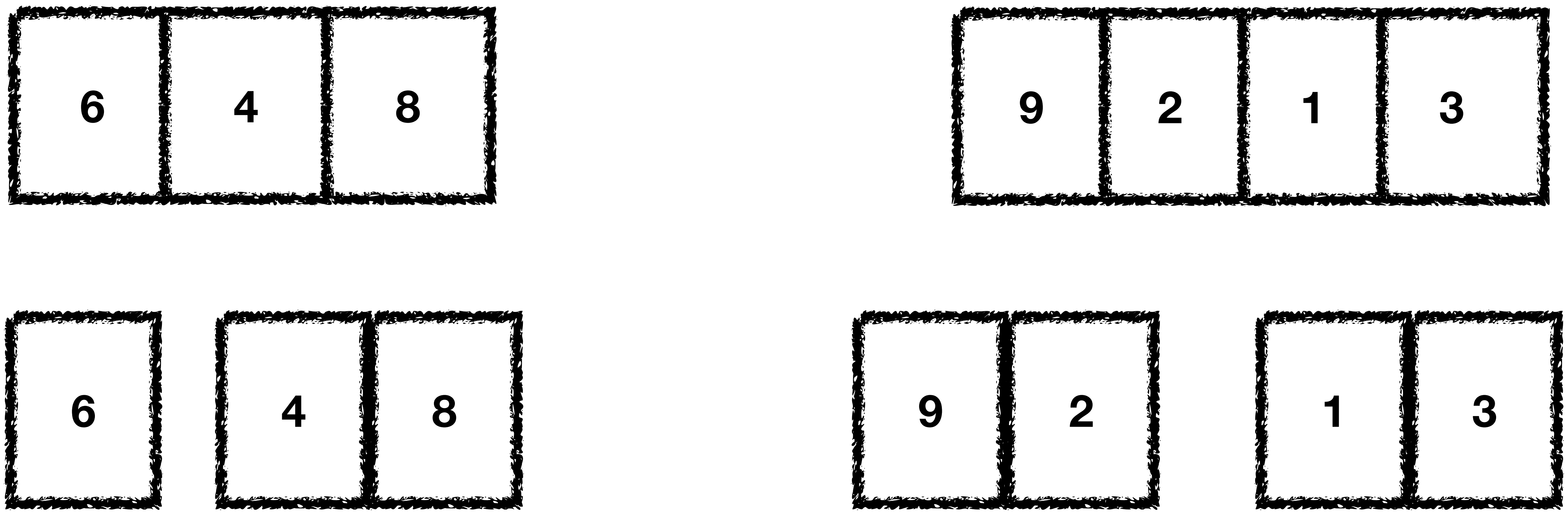

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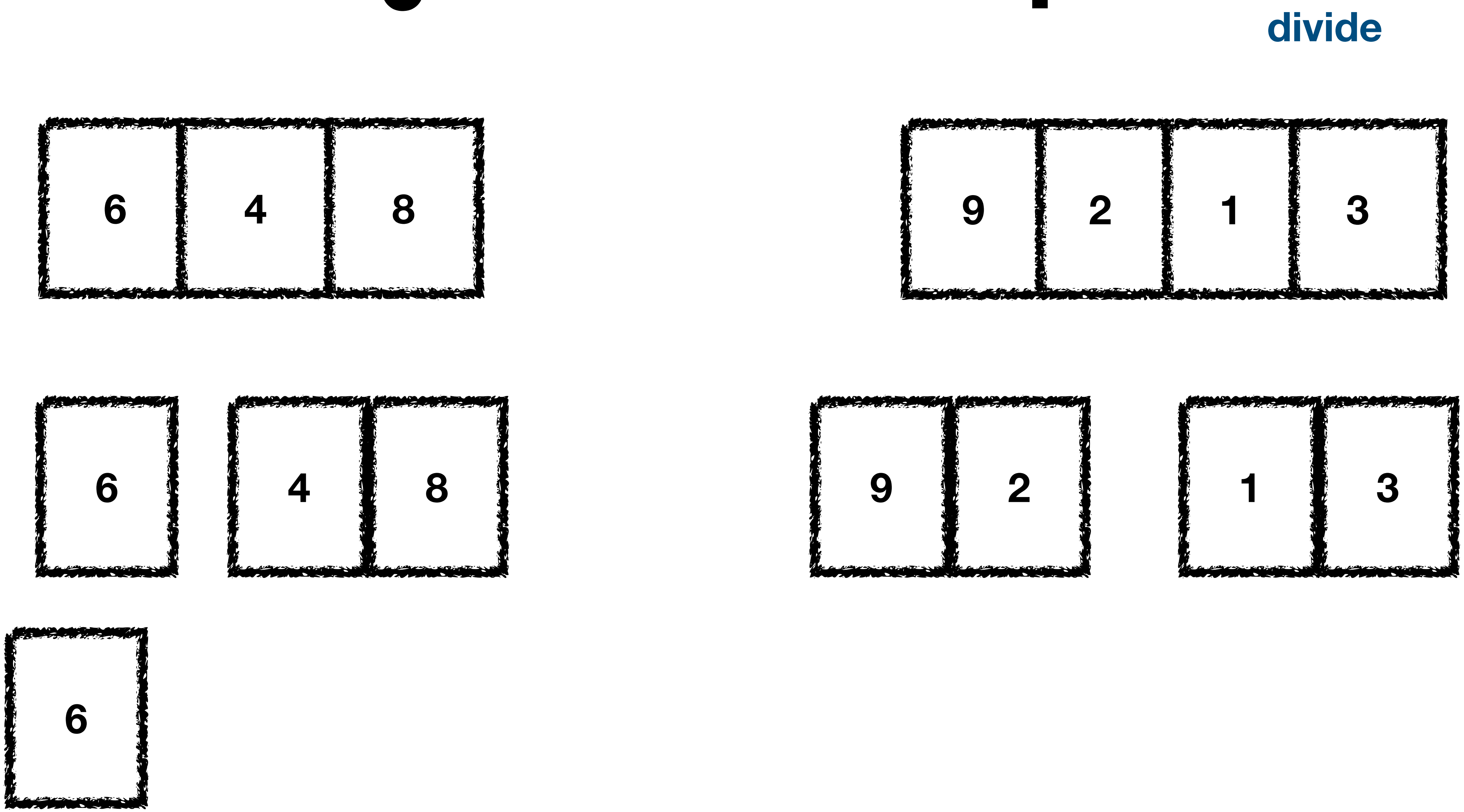

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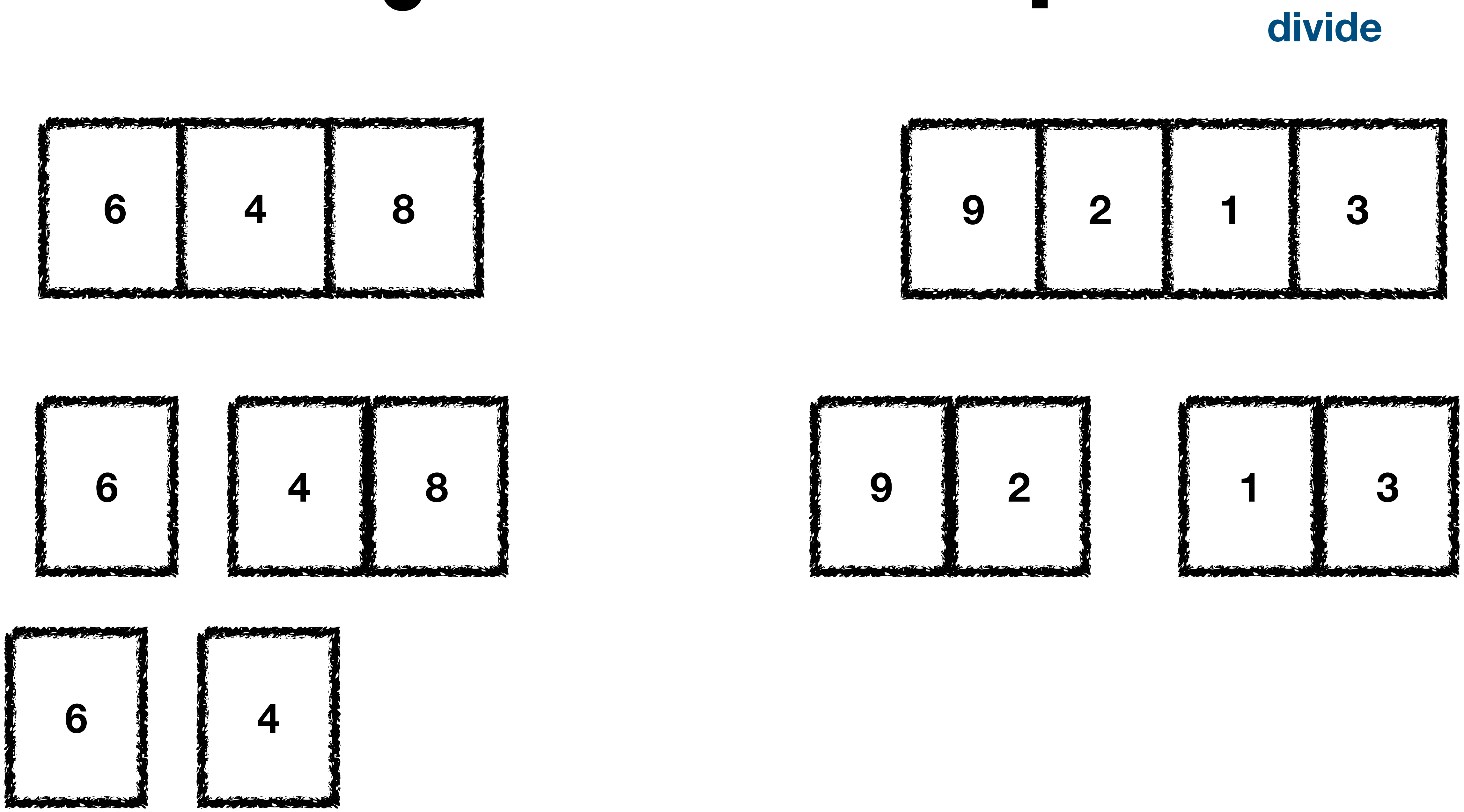

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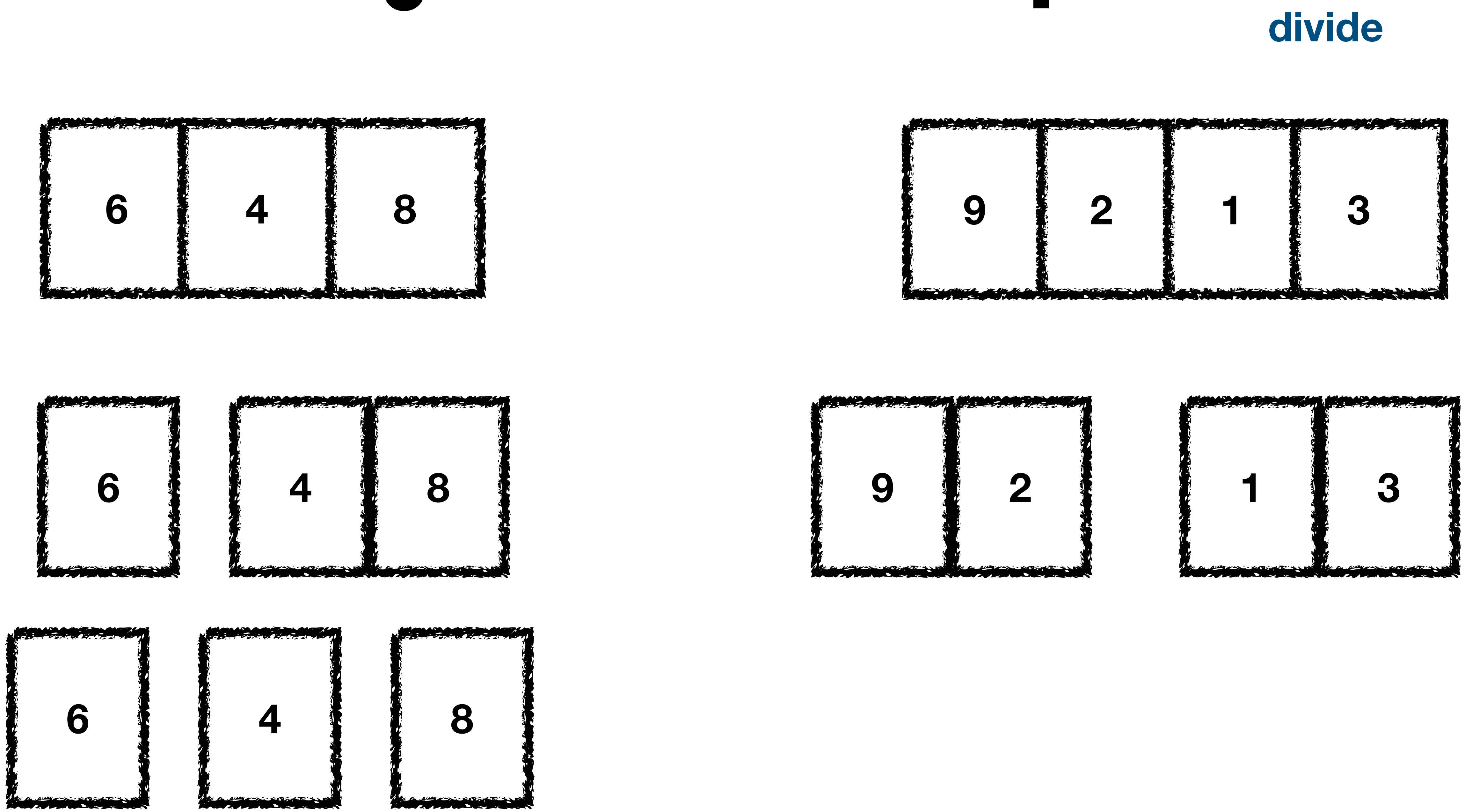

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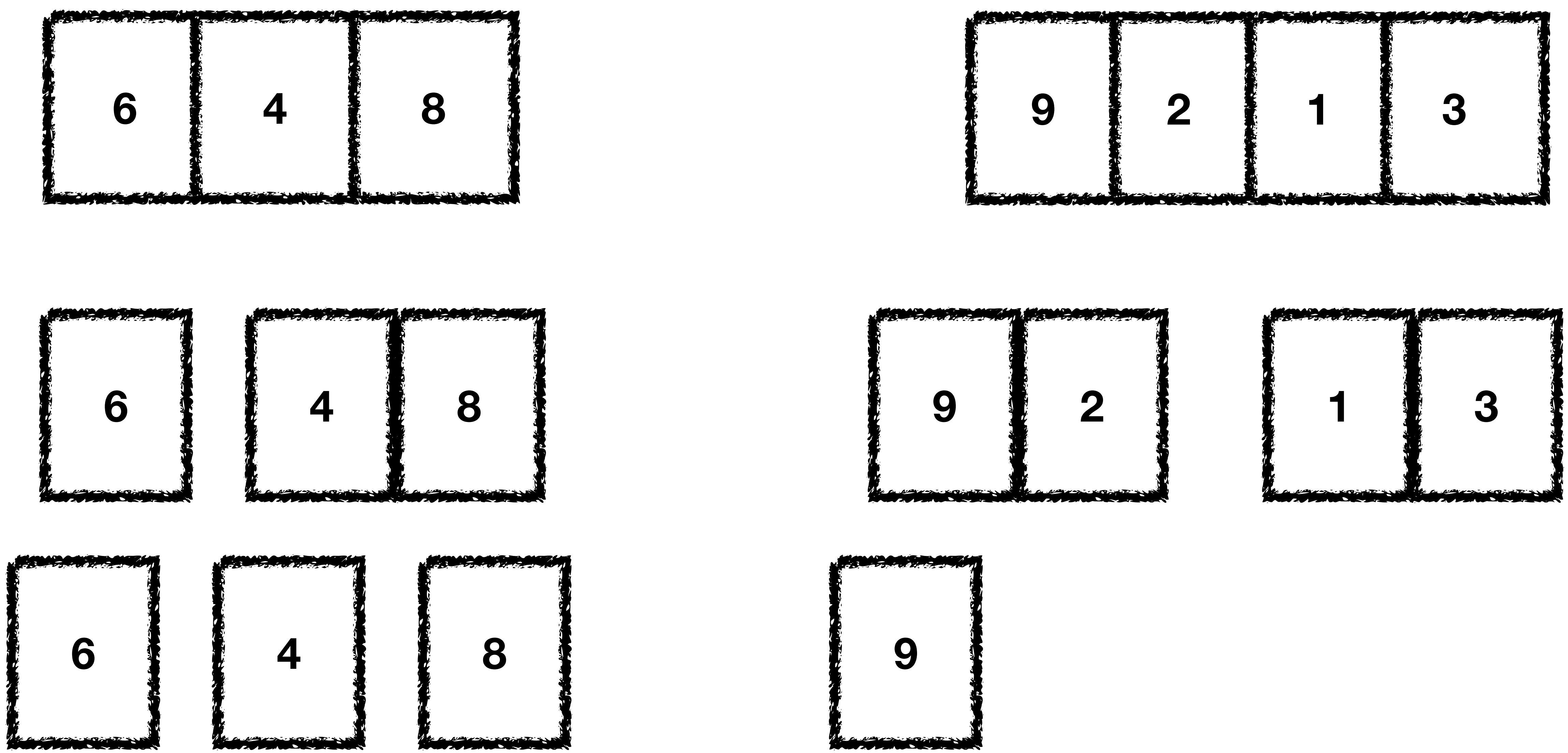

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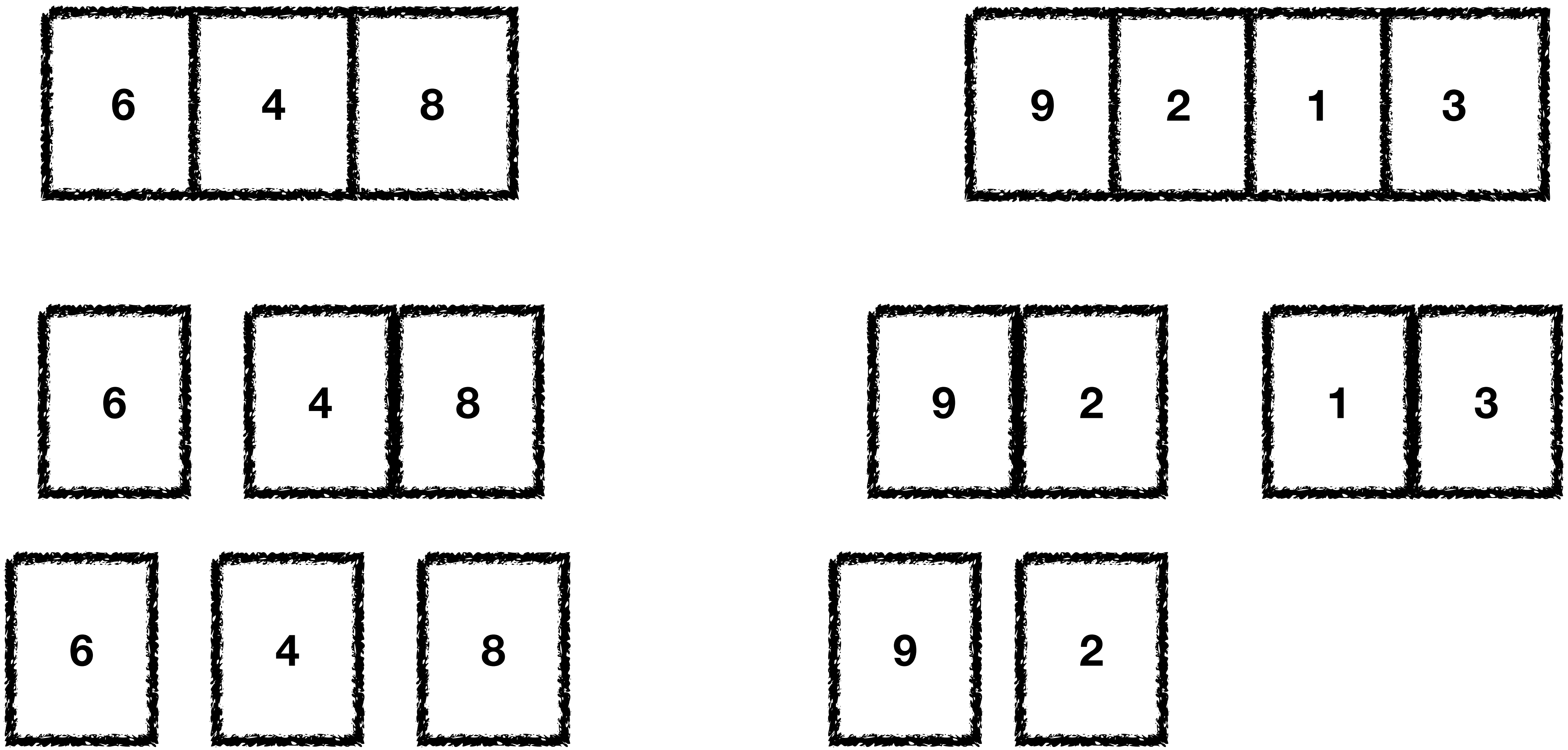

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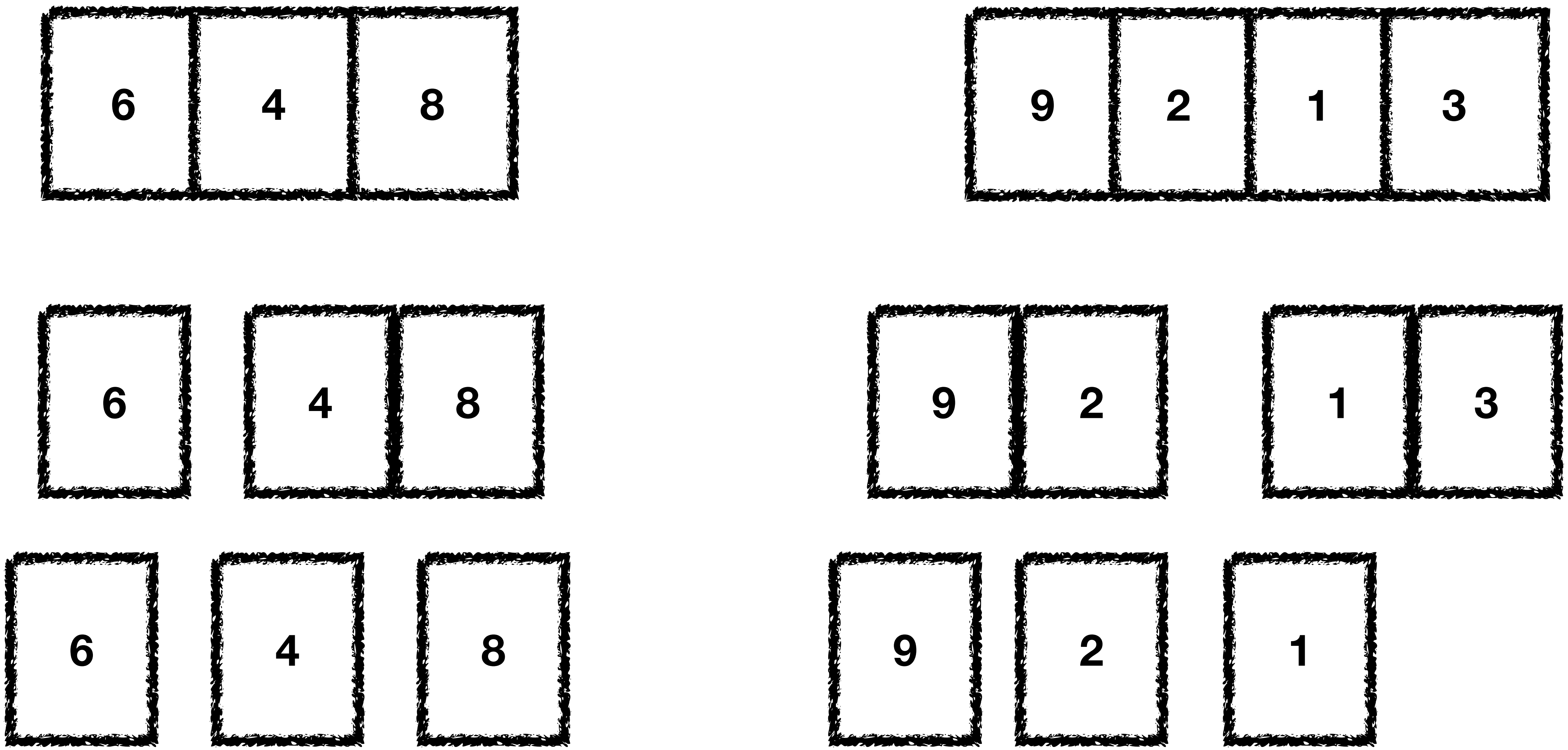

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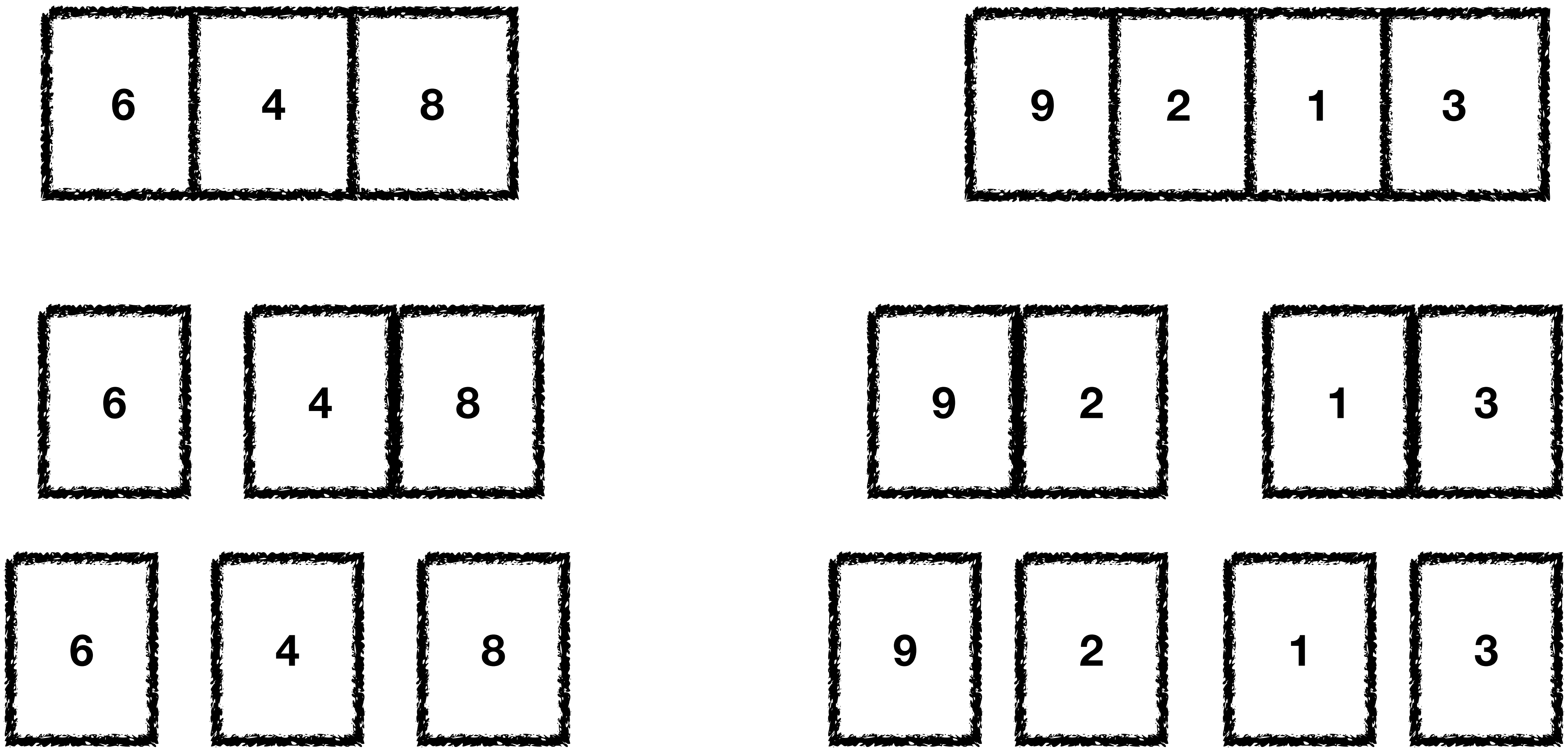
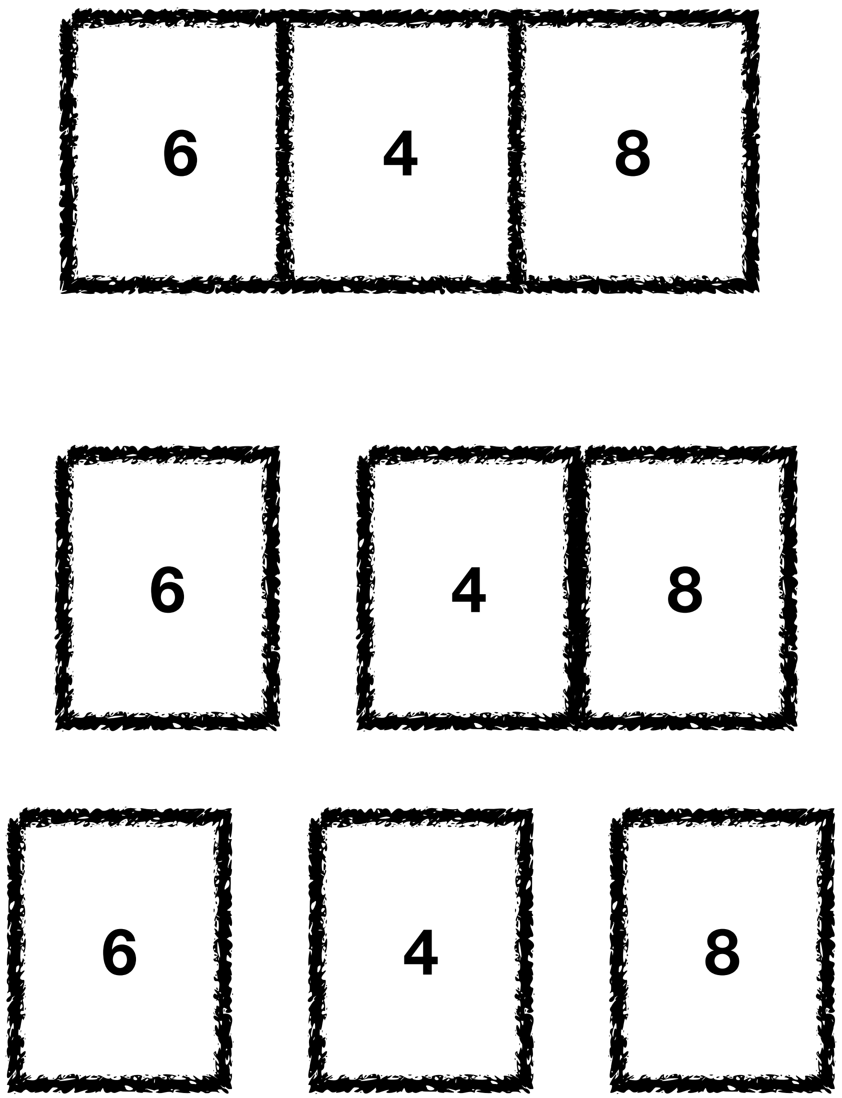


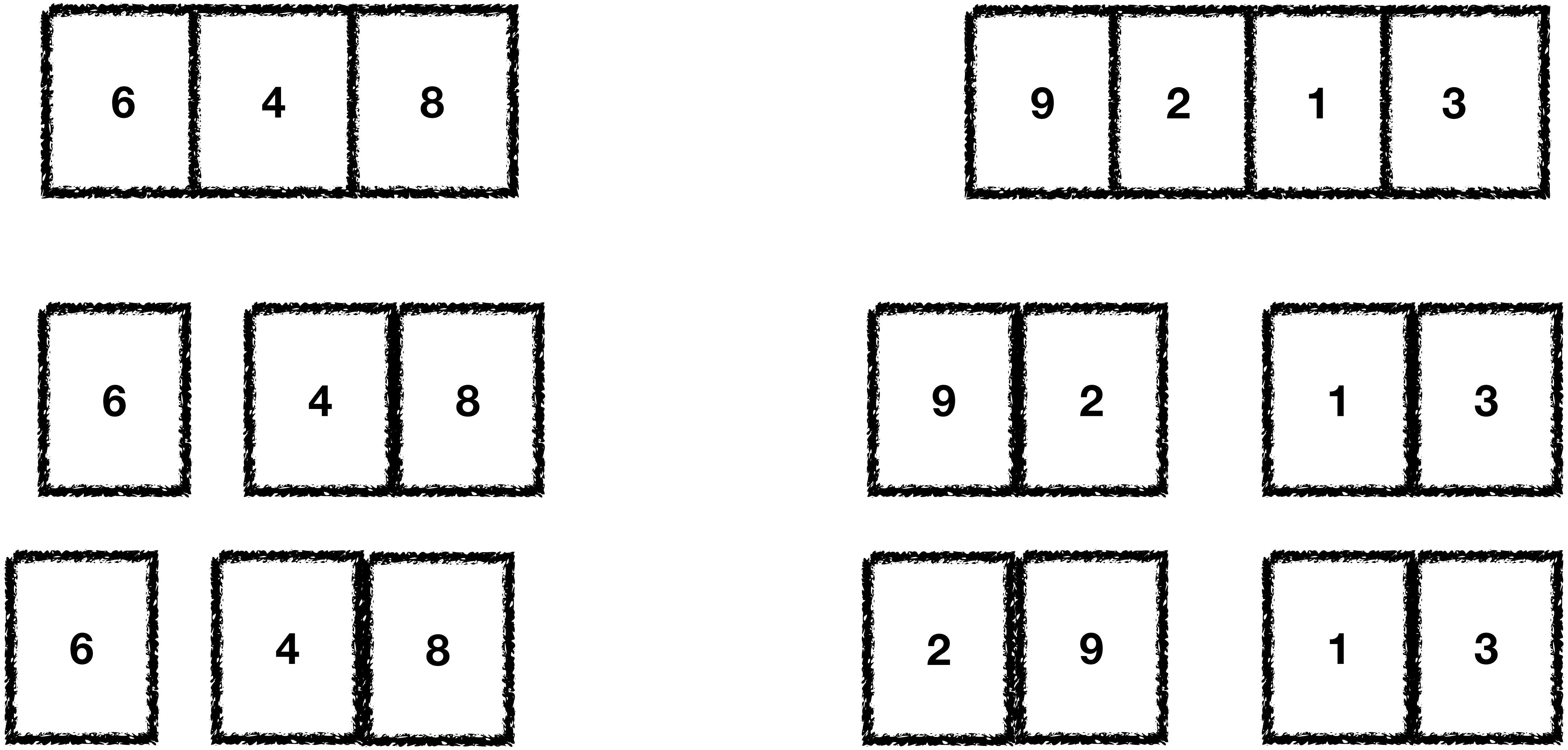
Mergesort example

divide merge



Mergesort example

divide merge



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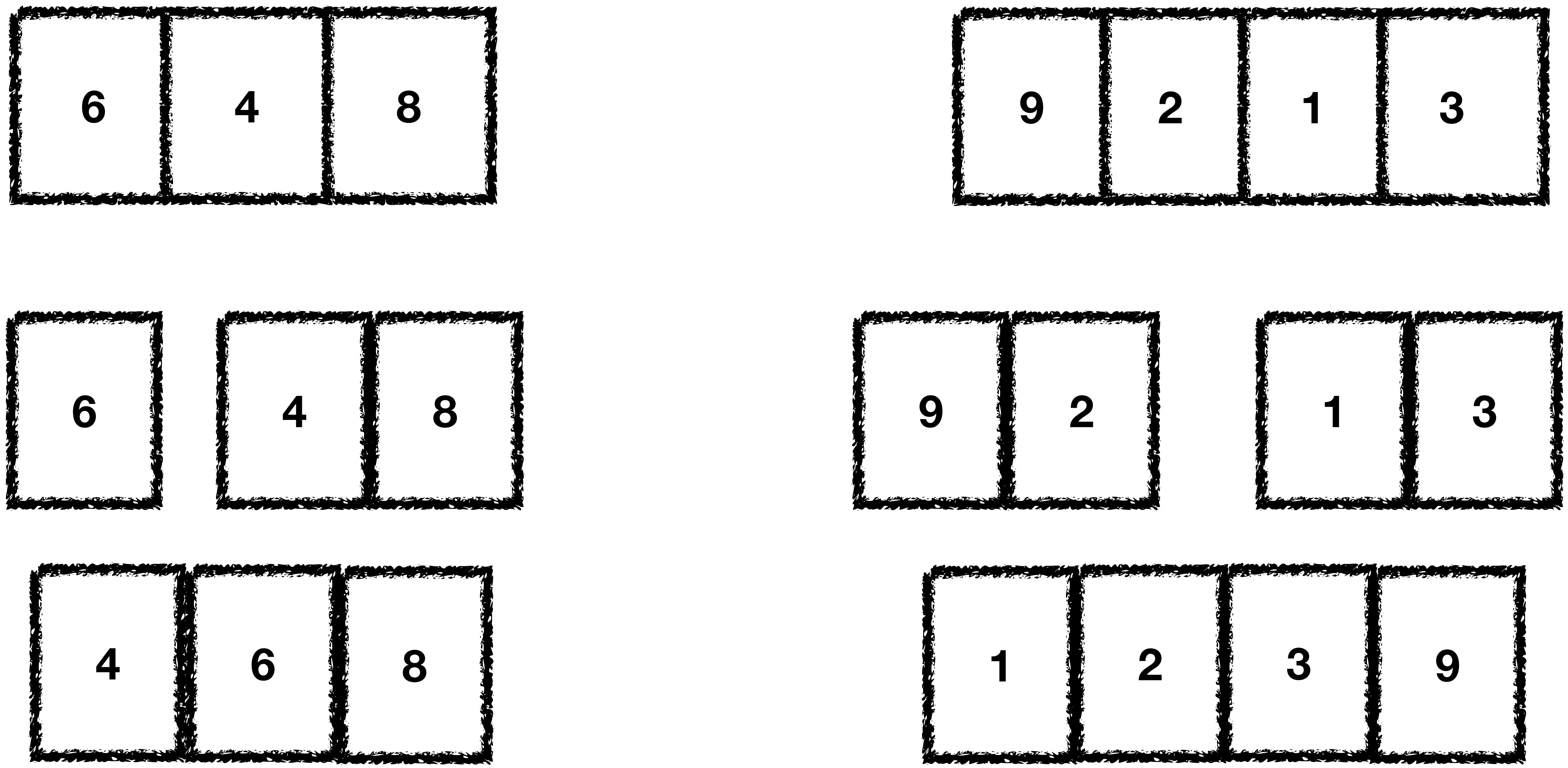

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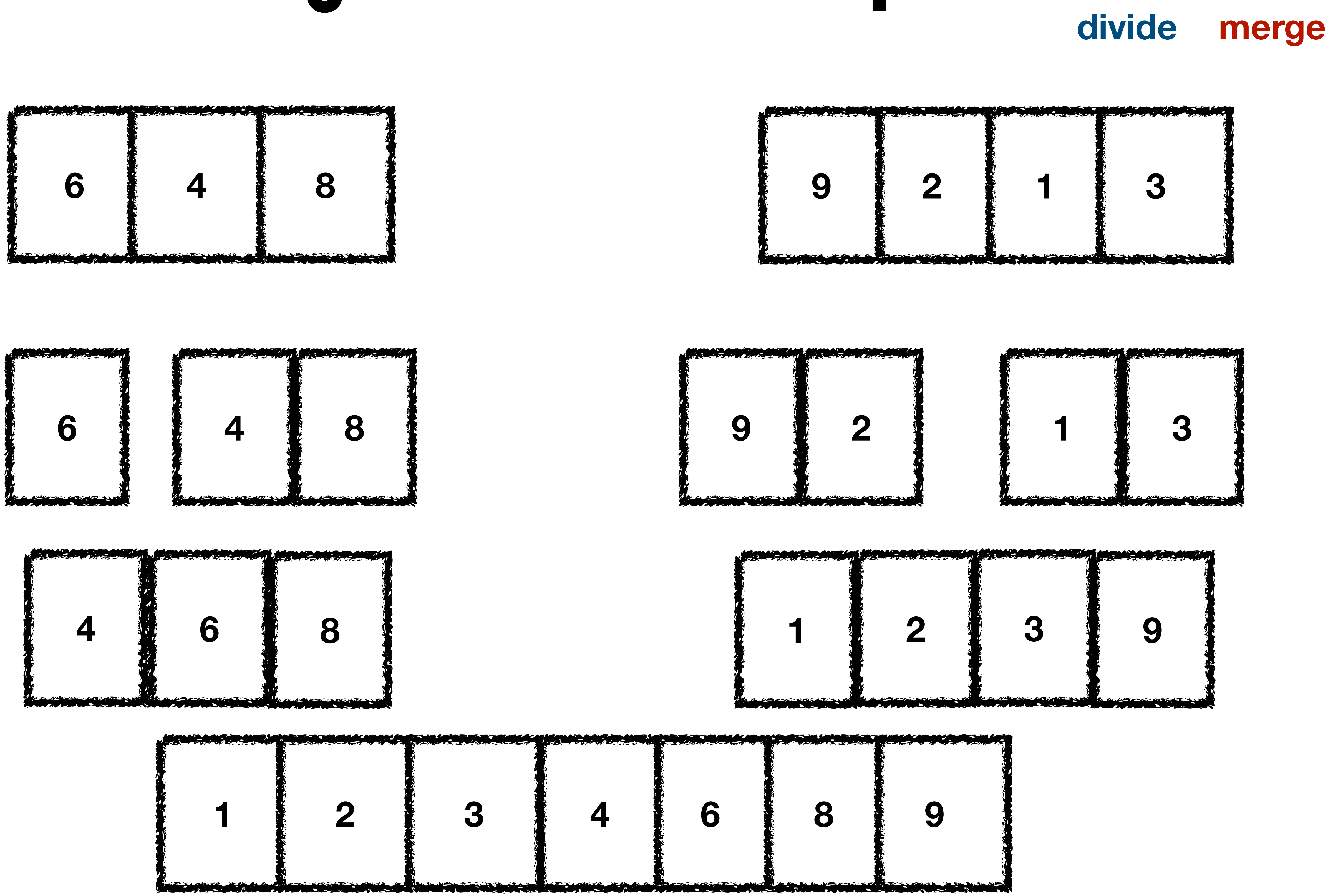

The Quicksort algorithm

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- Quicksort first divides the array into two parts, such that the first part is “smaller” than the second part.
    - This is done via the Partition procedure.
- Then it calls itself recursively.
- The two parts are joined, but this is trivial.

The Partition procedure

Procedure Partition(**A**[*i*,…,*j*])

Choose a **pivot element** **x** of **A**

k = *i*

For *h* = *i* to *j* do

If **A**[*h*] < **x**

Swap **A**[*k*] with **A**[*h*]
k = *k* + *1*

Swap **A**[*k*] with **A**[*h*]

Return *k*

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Correctness of **Partition**:
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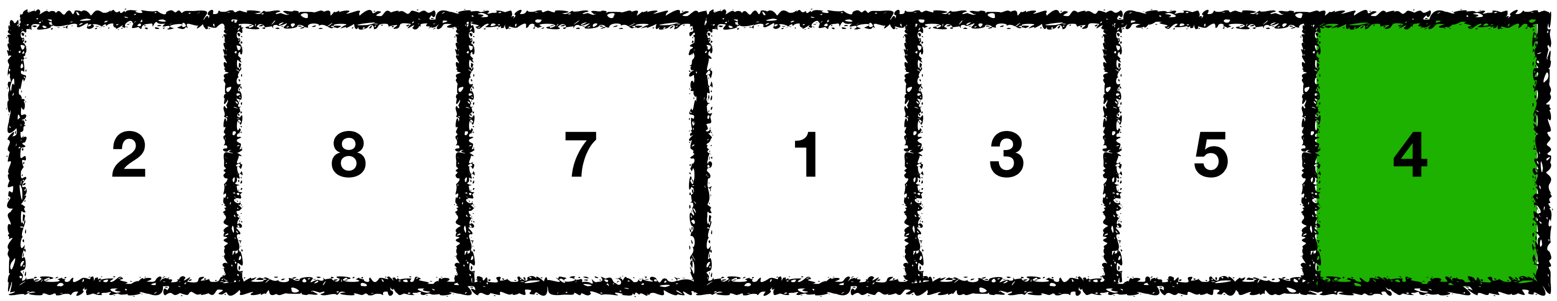
Running time **O(n)**

The Quicksort algorithm

2	8	7	1	3	5	4

The Quicksort algorithm



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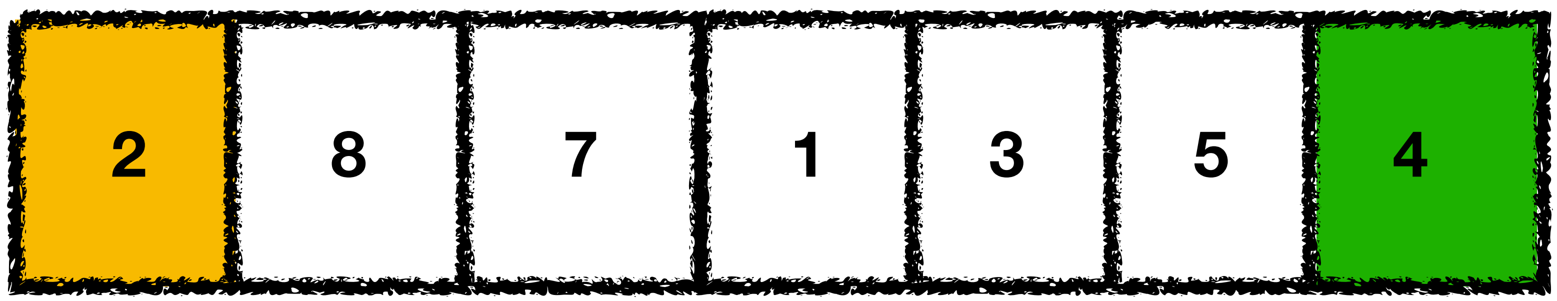

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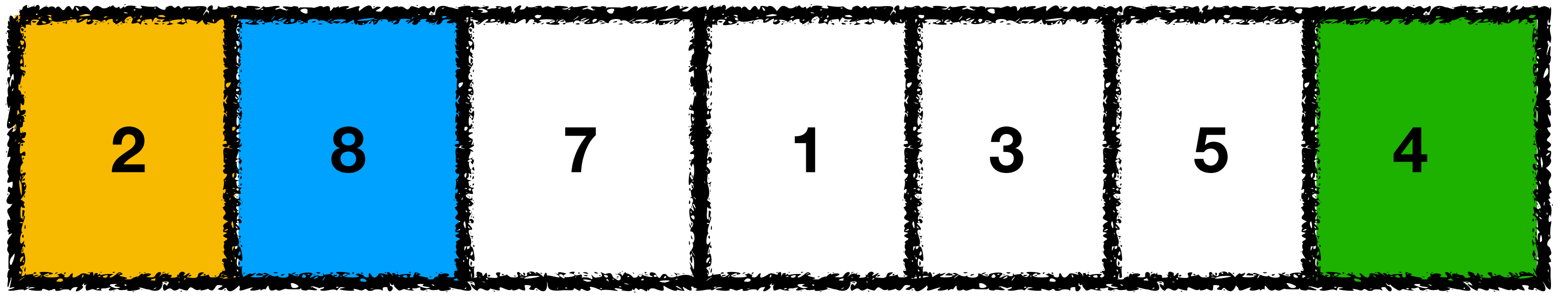

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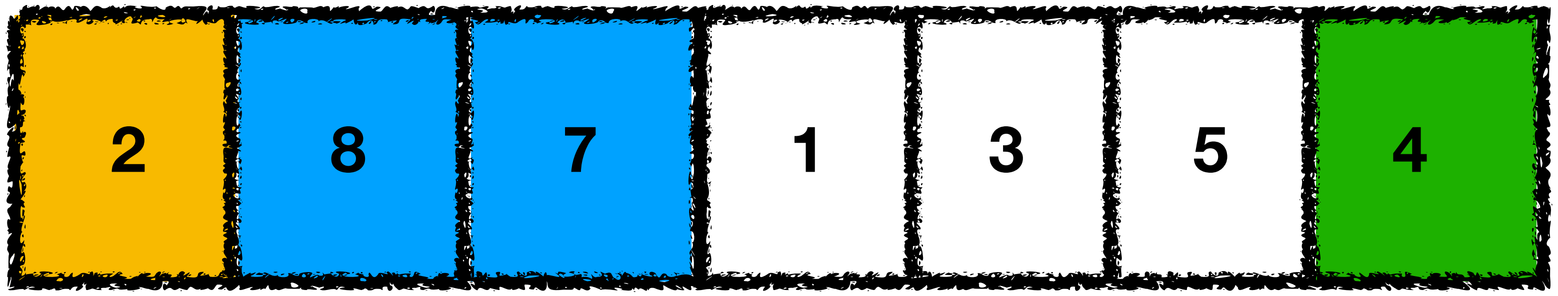

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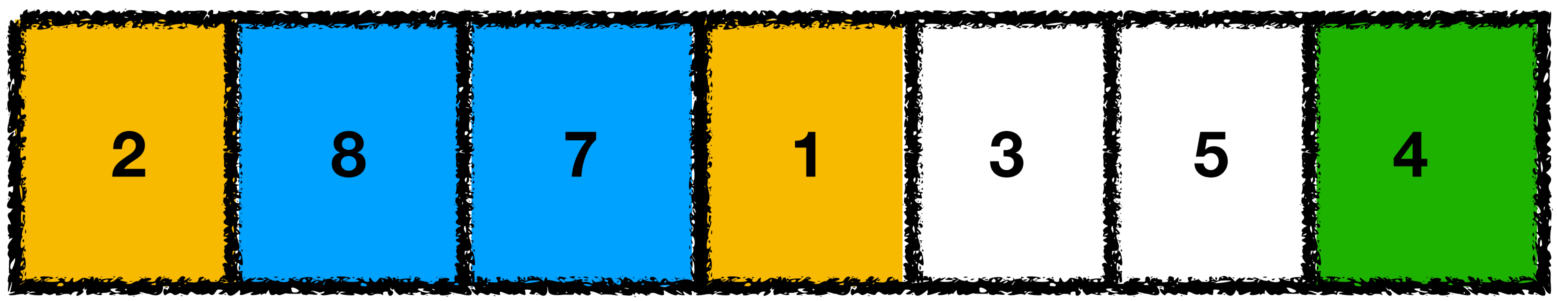

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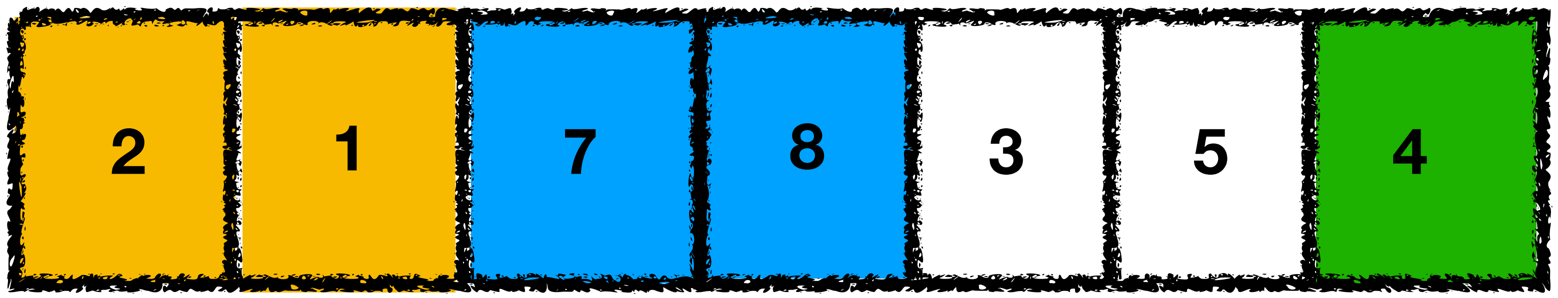

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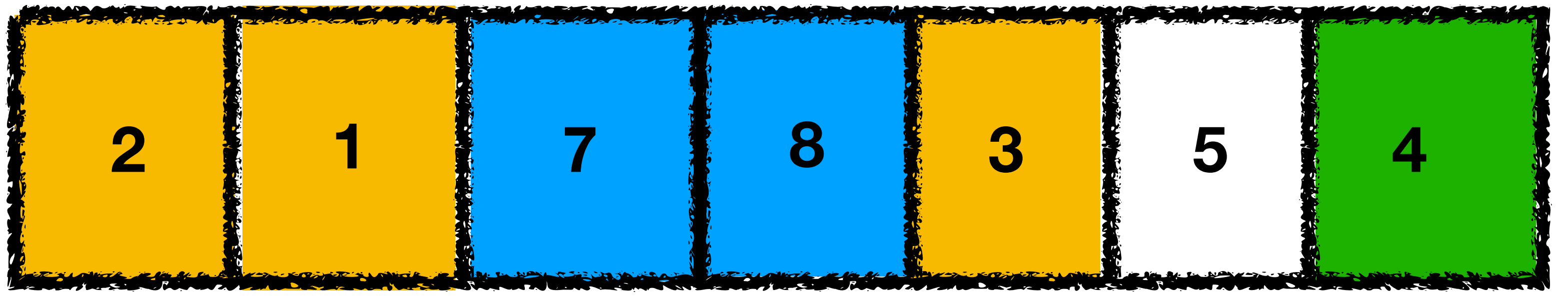

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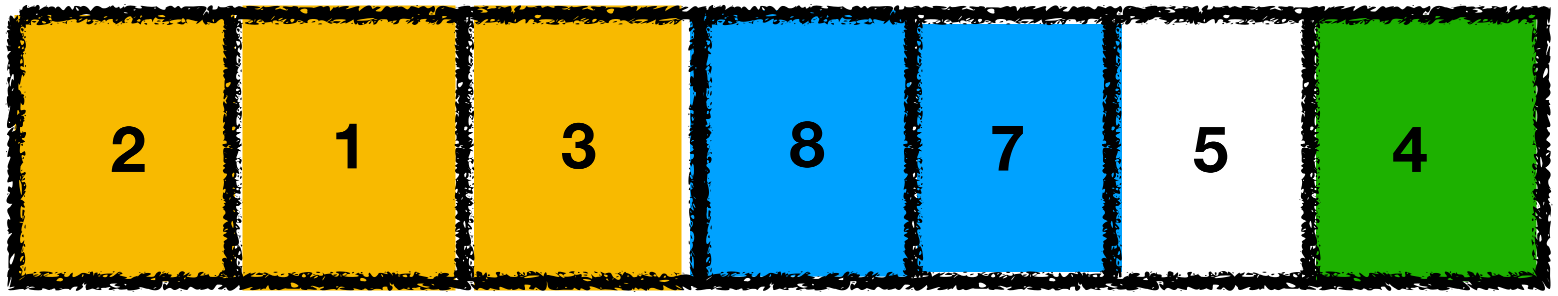

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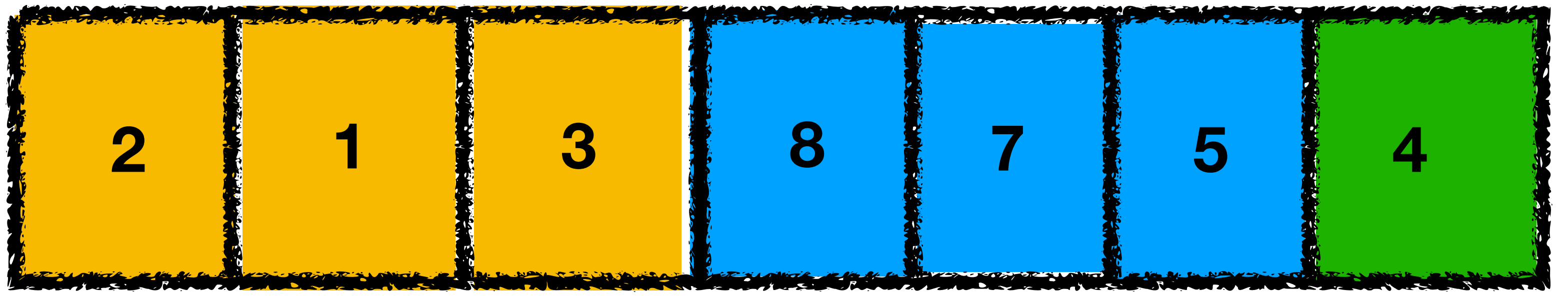

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Sort this using
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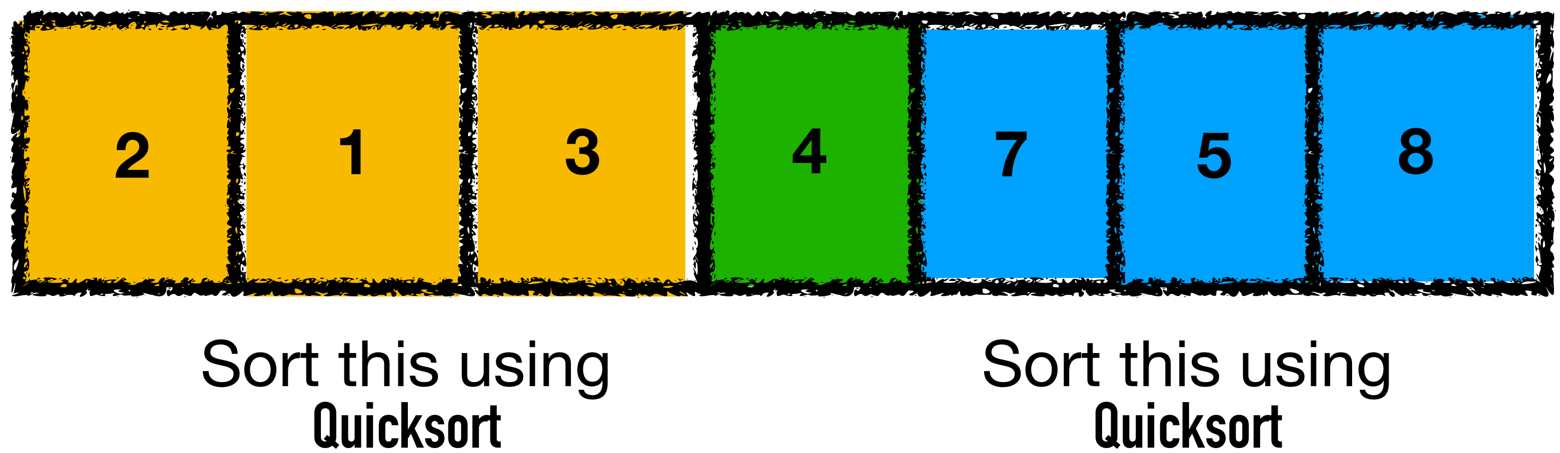

Algorithm **Quicksort**(**A**[$i,\ldots,j$])

y = **Partition**(**A**[$i,\ldots,j$])

Quicksort(**A**[$i,\ldots,y-1$])

Quicksort(**A**[$y+1,\ldots,j$])

Divide-and-Conquer

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- Split the input into smaller sub-instances.

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- Combine the solutions of the sub-instances into a solution for the problem.
- **Often:** For each sub-instance, the algorithm calls itself to solve it **(recursion).**

  The instances become so small that they can be solved via a **brute force** algorithm.

Worst-case Running Times

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What is the worst-case running time of **Mergesort**?

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$\Theta(n \lg n)$

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How do we prove these?

Worst-case Running Times, Upper Bounds

Algorithm **Mergesort**(**A**[$i,\ldots,j$])

If i=j, return i

q=(i+j)/2

$\mathbf{A}_{left}$=**Mergesort**(**A**[$i,\ldots,q$])

$\mathbf{A}_{right}$=**Mergesort**(**A**[$q+1,\ldots,n$])

return **Merge**($\mathbf{A}_{left}$, $\mathbf{A}_{right}$)

Worst-case Running Times, Upper Bounds

Algorithm **Mergesort**(**A**[$i,...,j$])

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Recurrence relation:

Worst-case Running Times, Upper Bounds

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Recurrence relation:

$$T(n) = 2T(n/2) + f(n)$$

Worst-case Running Times, Upper Bounds

Algorithm **Mergesort**(**A**[*i*,...,*j*])

If *i*=*j*, return *i*

q=(*i*+*j*)/2

$\mathbf{A}_{left}$=**Mergesort**(**A**[*i*,...,*q*])

$\mathbf{A}_{right}$=**Mergesort**(**A**[*q*+1,...,*n*])

return **Merge**($\mathbf{A}_{left}$, $\mathbf{A}_{right}$)

Recurrence relation:

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where $f(n) = O(n)$

Worst-case Running Times, Upper Bounds

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(next lecture)