Informatics 1 – Introduction to Computation
Computation and Logic

Julian Bradfield
based on materials by
Michael P. Fourman

Non-determinism
and Regular Expressions

Michael Rabin, 1931–

Dana Scott, 1932–

Another way of summing

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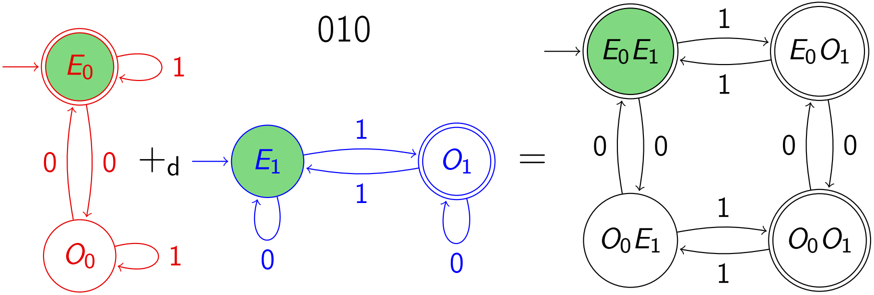

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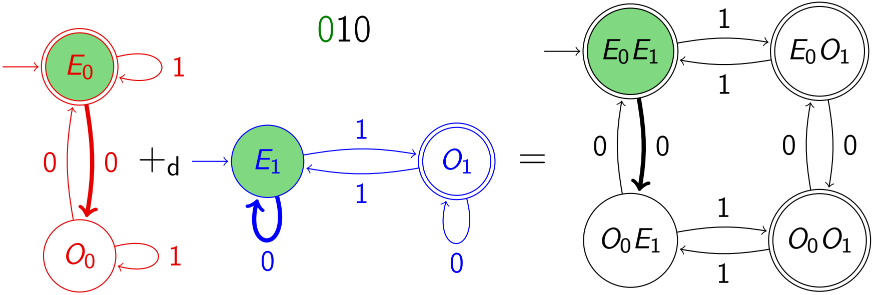

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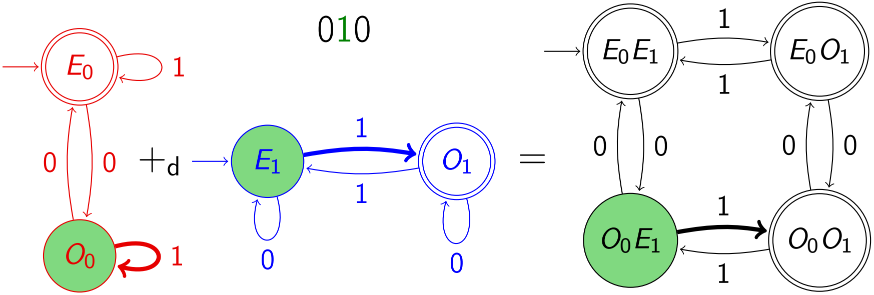

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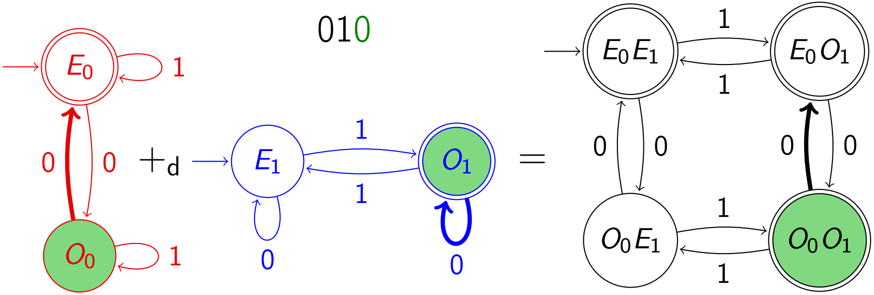

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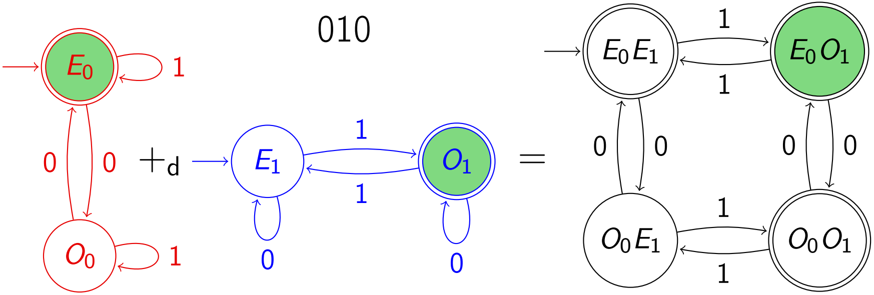

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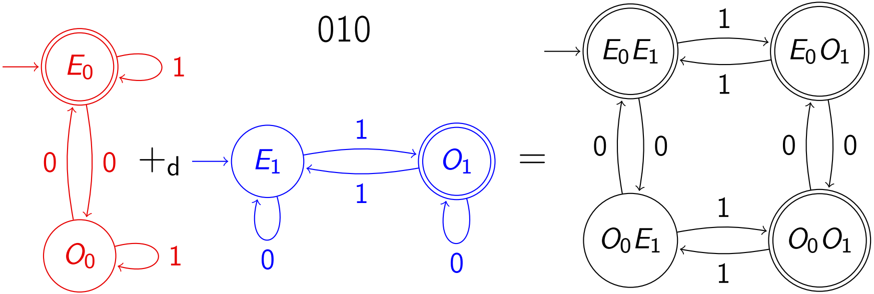

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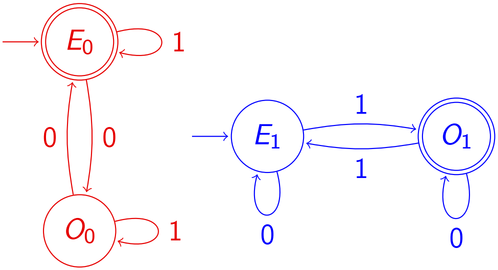

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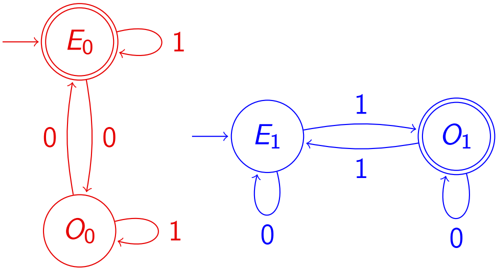

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But this isn't a DFA – so what is it?

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Note that we no longer need the black hole convention: we just omit unwanted transitions

The book chooses to define DFA like this with the added constraints 'S is a singleton' and 'δ is functional'. It's a matter of taste.

NFA behaviour – parallel understanding

An NFA may have many active/current states:

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One bit for each state. With a DFA we need $\lg n$ bits to track the single current state. So running an NFA requires exponentially more memory than a DFA.

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This notion of guess/choice is theoretical: it is not physically realizable. In particular, it is *not* probabilistic or chance choice, and it is *not* quantum anything.

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Another way to think is: if a magic oracle tells you which way to go at each choice, strings in the language are accepted. This way of thinking makes more sense at higher levels of complexity than FSMs.

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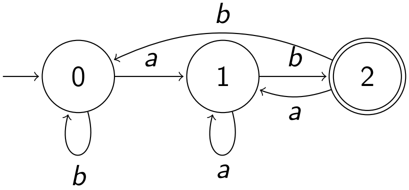

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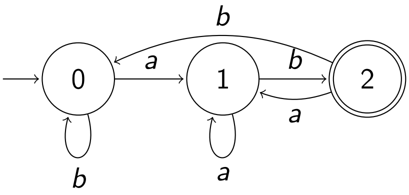
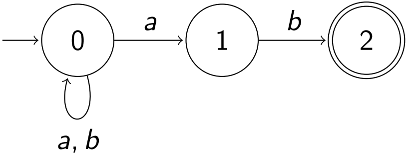


And here is an NFA:



More formalism

We extend some notations and terms to talk about NFAs:

- If $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation, then $\hat{\delta}\colon \wp(Q) \times \Sigma \to \wp(Q)$ is the state set transition function defined by $\hat{\delta}(\hat{Q}, a) = \bigcup_{q \in \hat{Q}} \{q' : \delta(q, a, q')\}$, and

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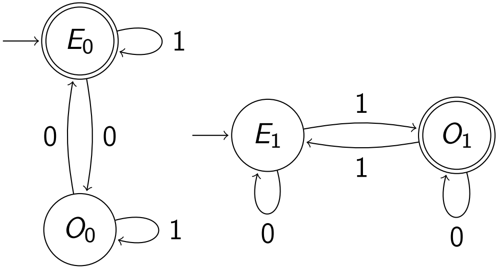
Theorem: For any NFA, we can build a DFA that accepts the same language.

And it's easy – in fact, we've already seen how it's done.

The subset construction

Here's an NFA with two start states:



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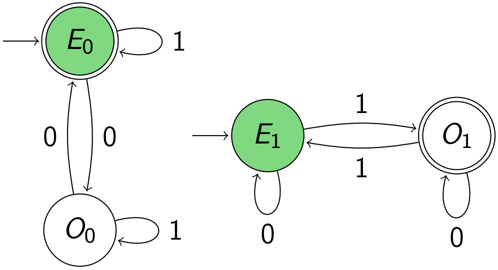

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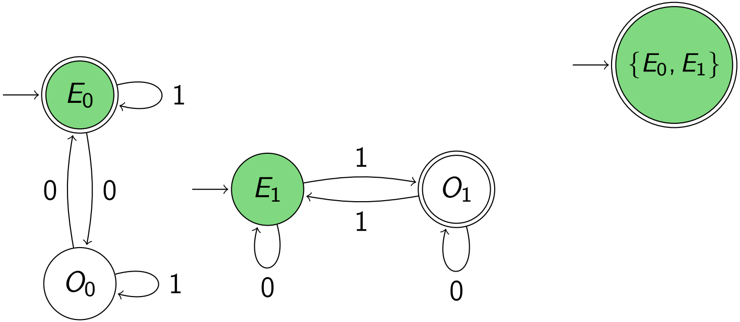

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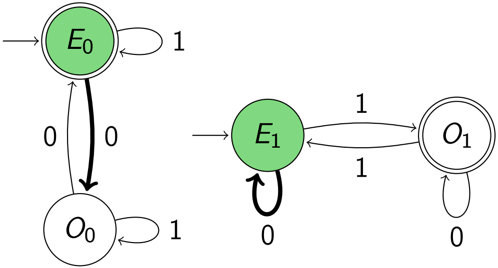

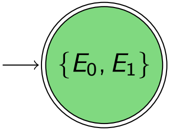

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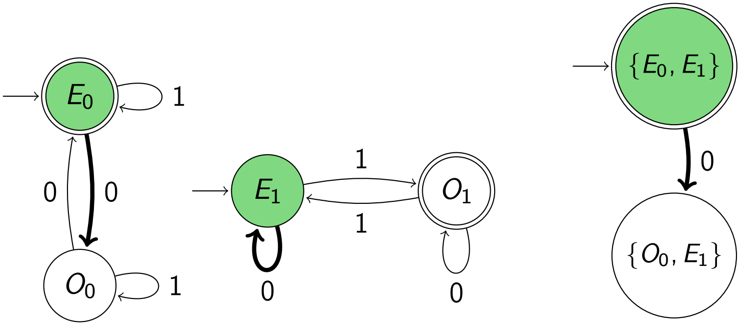

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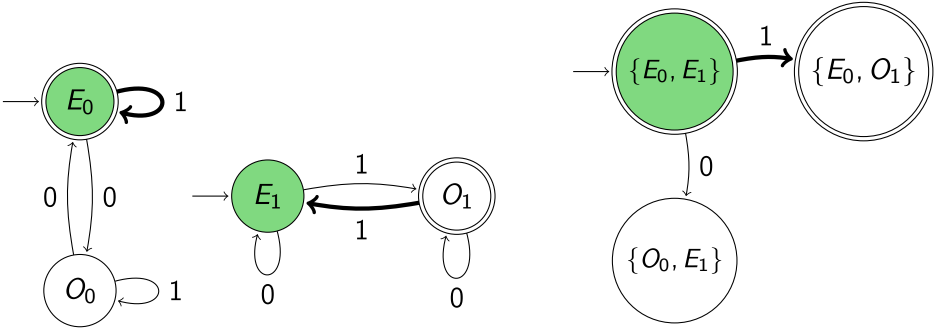

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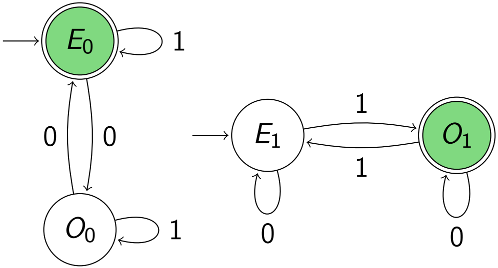

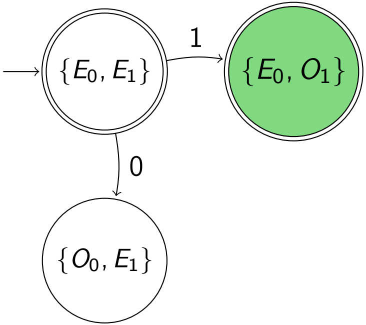

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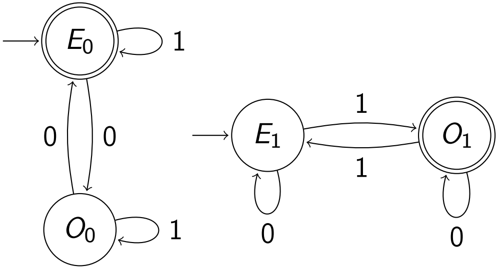

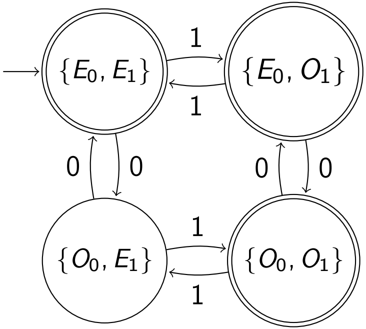

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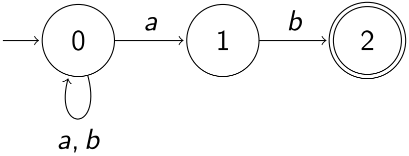
Another example from earlier, with non-det transitions:



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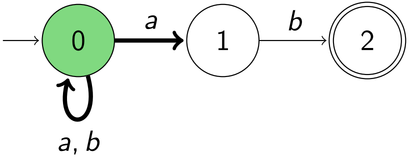

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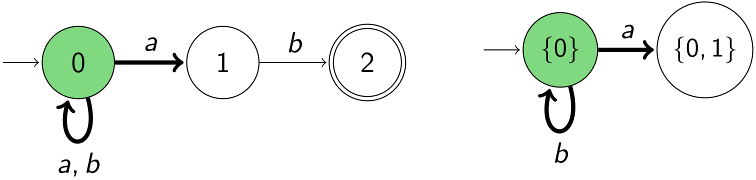

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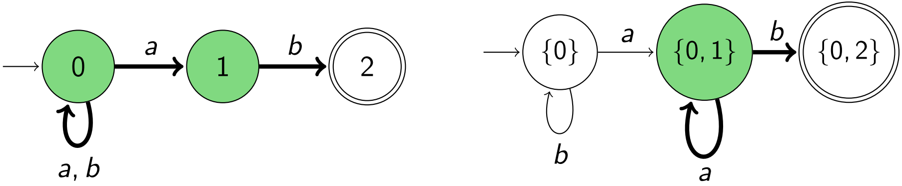

If *a* happens:

- There's no *a*-transition from 1, so 1 dies.
- But 0 stays active *and* (re-)activates 1.

If *b* happens, 0 stays active and the activity on 1 moves to 2.

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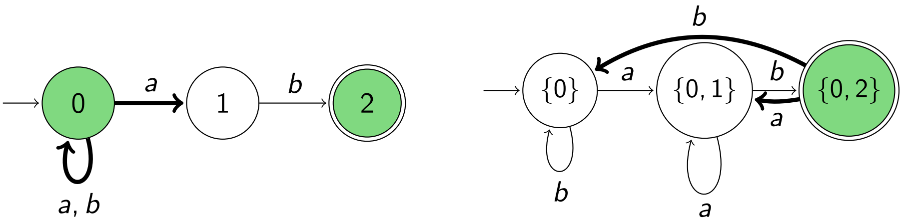

If *a* happens:

- There's no *a*-transition from 2, so it dies.
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If *b* happens:

- There's no *b*-transition from 2, so it dies.
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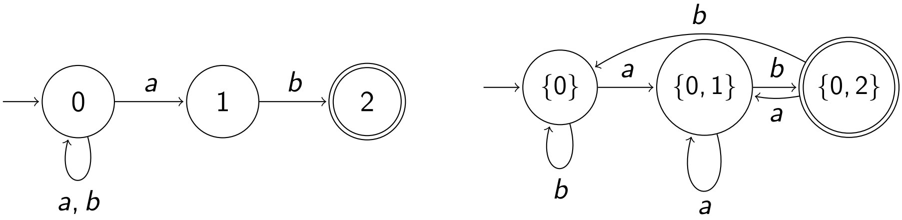

We have reconstructed the original DFA from slide 6! This is a happy coincidence.

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If we do it mathematically, at one fell swoop, it looks like this:
Given NFA $M = (Q, \Sigma, \delta, S, F)$, define DFA $\hat{M}$ by:

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- $\hat{\delta}$ is the state set transition function (slide 7) and
- $\mathscr{F} = \{Q' \subseteq Q : Q' \cap F \neq \varnothing\}$

In many cases, most of the superstates in $\wp(Q)$ can't be reached from the starting superstate S, so on-the-fly construction is almost always the right thing in practice.

Now convince yourself (using the book if necessary) that $L(M) = L(\hat{M})$.

It's always annoyed me that we have to write $A \cap B \neq \varnothing$ to say that A and B overlap. Somebody on reddit suggests $A \between B$. What do you think?

ϵ-NFAs

In practice, it's very useful to have a slightly extended notion of NFA.

An ϵ-NFA is an NFA which has an additional special symbol $\epsilon \notin \Sigma$, and a transition relation $\delta \subseteq Q \times (\Sigma \cup \{\epsilon\}) \times Q$.

If $q \xrightarrow{\epsilon} q'$, then the machine can move from q to q' *without reading any input*.

This makes it much easier to *concatenate* machines or build loops. (We'll see examples later.)

Everything we've done can be adjusted to ϵ-NFAs with a little work – see the book for details. In particular, the subset construction still works. (**Read** the book section on this, p. 331–3 on the draft pdf.)

The Greek letter lower-case epsilon has two common forms: standard ε and *lunate* ϵ. I like to use ε for the empty string, and ϵ for the silent transition, but that's just me ...

Building (ϵ-)NFAs: complement and product

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To complement an NFA, first convert to DFA and then complement: exponential blow-up in states.

Building (ϵ-)NFAs: sum

In compensation, the sum or union becomes much easier:

- Let $M = (Q, \Sigma, \delta, S, F)$, and $M' = (Q', \Sigma, \delta', S', F')$, *where* $Q \cap Q' = \varnothing$.
- The sum is $M + M' = (Q \cup Q', \Sigma, \delta \cup \delta', S \cup S', F \cup F')$.

In other words, just put the two automata side by side, as we did at the beginning of this week.

Some people write $M + M'$, some $M \cup M'$.

Building ϵ-NFAs: concatenation

The big win from ϵ is concatenation:

Given $L = L(M)$ and $L' = L(M')$, can we build a machine that accepts $LL' = \{ss' : s \in L, s' \in L'\}$?

Hence we know regular languages are closed under concatenation.

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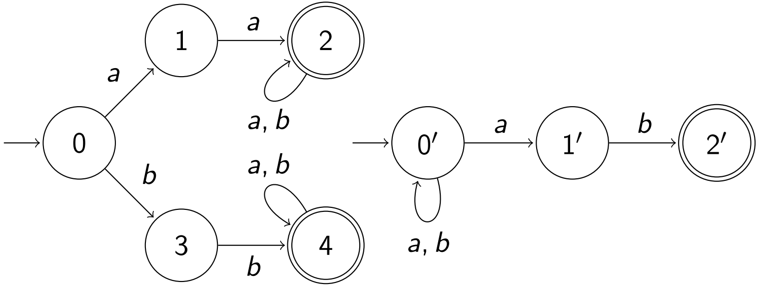

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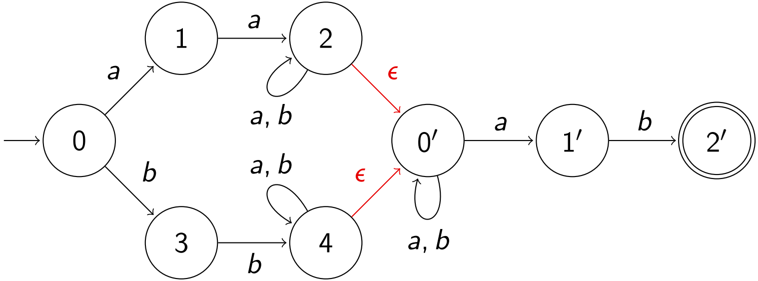
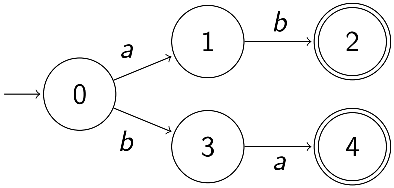


Building ϵ-NFAs: looping

A generalization of concatenation is concatenating a machine *with itself*:



Note that the initial and final states remain such.

This machine accepts $\{ab, ba\}$.

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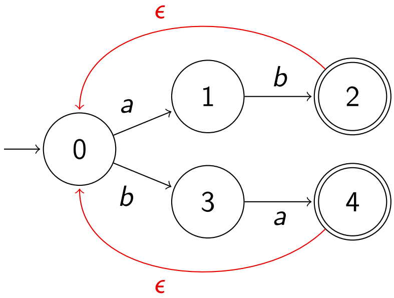

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This machine accepts strings made up of sequences of *ab* and *ba*.

Regular Expressions

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Regular expressions (regexes, regexps) are a simple and universally used way of describing string languages. Any program that does anything with text probably uses them.

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If $L \subseteq \Sigma^*$, L^* means $\bigcup_{n>=0} L^n$, where $L^0 = \{\varepsilon\}$.

Regexp examples

- $(ab|ba)^*$ is the language from slide 16.2.
- $(aa|bb)(a|b)^*ab$ is the language from slide 15.4
- $1^*(1^*01^*01^*)^*$ is the language of strings with an even number of 0s. (Why? Why is this so much more complex than the DFA from last week?)

Regular expressions to ϵ-NFAs

The constructors for regexps are exactly the operators with easy ϵ-NFA constructions. So it is very easy to convert regexps to ϵ-NFAs, starting with the automata for ε and a.

So easy it's not even worth having a slide!

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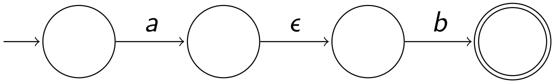

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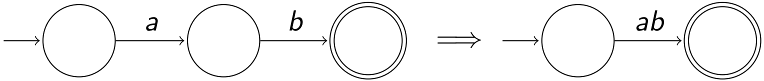

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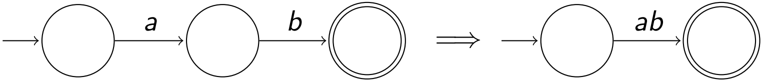
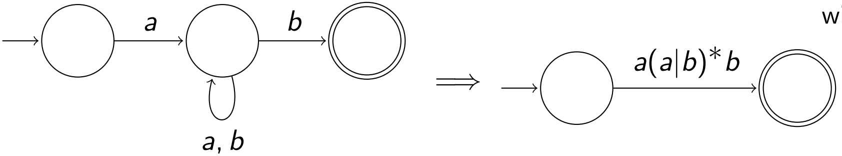


If the state to be removed has self loops, that's still easy:



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We've combined multiple (a, b) transitions into one with $|$.

ϵ-NFAs to regular expressions, cont.

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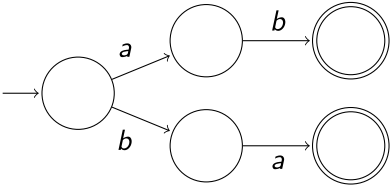

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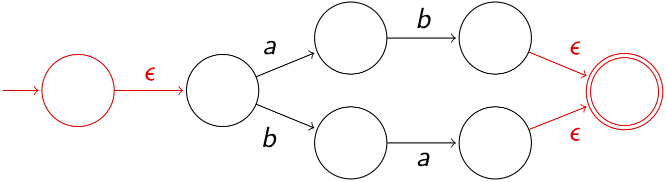

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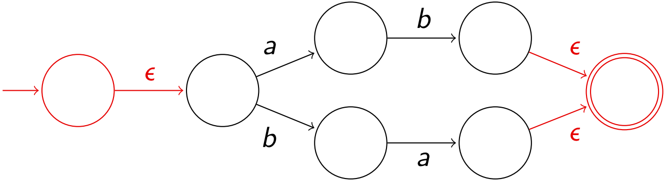

Now repeating the previous procedure will bring you to a big | regexp on one transition between the red states.

Hence: regular expressions describe exactly the regular languages.

The book describes a different method, solving equations and using *Arden's Rule*. It is essentially equivalent, and perhaps easier to program, but less easy to understand intuitively.

If you are feeling strong, try our technique on the 'even 0s and odd 1s' machine from last week.

Regexps in real life

Actual regexps have many more constructors, to make them easier to use. Some examples:

- *Character classes* $[abc]$ meaning $(a|b|c)$, and *ranges* $[a\text{–}f]$ meaning $(a|b|c|d|e|f)$.

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and many more. The older ones are syntactic sugar, but modern languages may add constructors that are no longer regular.

'Syntactic sugar' refers to syntax that doesn't increase the power of a language, but makes it easier and shorter to write.

Matching regexps in real life

Usually, programming languages assume you want to see if R matches any *substring* of input s. I.e. R implicitly means $(.^*R.^*)$.
To avoid this, you can *anchor* to the beginning and/or end of s using $\hat{}R\$$.

The language Perl introduced extremely powerful 'regexps', which have been taken up by other language as 'PCRE's. Perl's regexps are not at all regular. I have an entire talk about them!

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So given input $s = aaa$, and $R = $ ^$(a^*)(a^*)$\$, which bits of s are matched by the two parenthesized parts? (Remember NFAs are non-deterministic!)

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Programming languages *determinize* regexps: they say that * is *greedy*, i.e. matches as much as possible. So s would be matched as $(aaa)()$.

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